Algebraic Geometry and Group Theory in Geometric Constraint Satisfaction

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Abstract

The determination of a set of geometric entities that satisfy a series of geometric relations (constraints) constitutes the Geometric Constraint Satisfaction or Scene Feasibility (GCS/SF) problem. This problem appears in different forms in Assembly Planning, Constraint Driven Design, Computer Vision, etc. Its solution is related to the existence of roots to systems of polynomial equations. Previous attempts using exclusively numerical (geometry) or symbolic (topology) solutions for this problem present shortcomings regarding characterization of solution space, incapability to deal with geometric and topological inconsistencies, and very high computational expenses. In this investigation Grobner Bases are used for the characterization of the algebraic variety of the ideal generated by the set of polynomials. Properties of Grobner Bases provide a theoretical framework responding to questions about consistency, ambiguity, and dimension of the solution space. It also allows for the integration of geometric and topological reasoning. The high computational cost of Buchberger's algorithm for the Grobner Basis is compensated by the choice of a non redundant set of variables, determined by the characterization of constraints based on the subgroups of the group of Euclidean displacements $SE(3)$. Examples have shown the advantage of using group based variables. One of those examples is discussed.

1 Introduction

Spatial Geometry relations are of primary importance in Computer Aided Design, Manufacture and Process Planning. In particular, Spatial Reasoning applies in diverse areas such as Assembly Planning, Computer Vision, Robotics, Constraint Driven Design and Drafting, Feature Extraction, Machine Tool Selection and Specification, etc. In Spatial Reasoning, two main areas can be identified; *Static* and *Dynamic* Reasoning. *Static* Reasoning problems are those concerned with fully and unambiguously defined entities. Typical problems include *boolean* queries testing a particular relation among entities and *construction* queries which create entities satisfying relations with other given entities in the world. These problems have well defined (although not unique) answers. A taxonomy of the problems in flat entity worlds was compiled and developed in [19]. In this taxonomy, the problems are divided into Convex Hulls, Inclusion, Intersection, and Closest Point problems. In *Dynamic Reasoning*, a set of geometric entities is specified by geometric relations (also called *constraints*) in the context of a given world. As the set of relations varies, the position (and the very nature) of the entities in the space changes. The specification may be ambiguous, resulting in an infinite number of possible answers, or inconsistent, producing an empty solution space.

The GCS/SF problem in a flat entity world can be stated as follows: Let a World W be a closed, homogeneous subset of E^3, and a set of zero curvature geometric entities $S = \{e1, ..en\}$ (points, straight lines, planes). A set of spatial relations between pairs of entities $R = \{R_{i,j,k}\}$ are specified, where $R_{i,j,k}$ is the k^{th} relation between entities i and j. The goal is to find *a* position for each entity e_i in the world W, $R_{i,W}$, consistent with all relations R specified on it. The world W is a basic, fixed scenario in which the entities S satisfying R will eventually be instantiated. One says that *S is feasible for W and R*, and denotes this fact by $S = feasible(W, R)$. This problem can be translated into the solution of a set of polynomial equations $F = \{f_1, f_2, ...f_n\}$; therefore *F is the polynomial form of the problem (W,R)*; and it is written as $F = poly\_form(W, R)$. If *S is a solution for F*, we write it as: $S = solution(F)$.

A numerical approach to the solution of a polynomial set of equations for geometric purposes has been investigated in [17, 13, 1, 6]. Numerical solutions, besides their problems with convergence only solve the geometric part of the problem; i.e. they produce a feasible configuration if it exists and the algorithm converges. They *sample* the solution space at a single point, while the set of questions to be answered includes the degrees of freedom still available, the consistency of the set of constraints, the existence of possible redundancies, etc. Therefore, even in the event of perfect convergency, numerical solutions provide only a partial answer to the problem.

The algebraic variety of the ideal generated by a given polynomial set is the set of common roots of the polynomials. Algebraic Geometry allows the characterization of the algebraic variety (multiplicity of solutions, dimension of solution space, etc). It is, from the theoretical point of view, a well defined problem, whose solution can be obtained by several techniques in algebraic geometry such as Grobner Basis [10, 5, 16] and methods of resultants [11, 14, 15]. A

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related subproblem, automatic proof of geometric theorems can be also solved by the Characteristic Sets method [8, 7]. The practical disadvantage of the algebraic geometry methods is the large computational expense incurred in symbolically calculating the basis for the ideal generated by the polynomial set [11]. These techniques neglect the underlying geometric meaning of the polynomials forming the set; the manipulation of variables or polynomials is dictated by algebraic techniques, not by the physical meaning of the variables.

A symbolic, degree of freedom analysis for the problem has been devised for kinematics by using group theory [9, 2, 3]. By expressing the kinematic constraints in terms of the subgroups of the Euclidean Group $SE(3)$,[1] the simultaneous enforcement of constraints can be expressed as intersection of the sets conforming the corresponding subgroups, and the composition of constraints can be characterized as the product of the corresponding sets using the group operation [12, 9, 2, 3]. This method presents the advantage of expressing the problem in terms of physically meaningful and independent variables, therefore stressing the geometric origin of the problem. It is limited however, in the sense that it does not deal with geometric inconsistencies, and it only considers constraints formed by single subgroups of $SE(3)$ (also called **trivial**), owing to the fact that (group) multiplication of subgroups is not a closed operation, i.e. the multiplication of two subgroups in general is not a subgroup [21, 23, 6].

In [20], Sturmfels treats the calculation of Grobner Basis under finite group action. Grobner Bases are calculated for (the ideal generated by) a set of polynomials F on a ring $C[x_1, ..x_n]$ in which the variables x_i can be permutated in the polynomials without changing F. That means, the set F is invariant under the action of a *finite* group Γ. Sturmfels reports large computational expenses in calculating instances of Grobner Basis for Ideals generated by symmetric sets of polynomials, and proposes an algorithm to calculate Grobner Bases for symmetric input data F, which circumvents some of the initial problems. Although the present article is not explicitly concerned with *finite* groups, it has been observed in our work that many scenarios do produce sets of relative position matrices which essentially permutate the input variables. Therefore, a continuation of this work might include the investigation of influence of symmetries on the performance of Grobner Basis algorithms, and the ways to avoid expensive computations.

In this work([18]), an integration of methods of algebraic geometry with formalization from group theory will be undertaken, to exploit the particular strengths of each area, and minimize their disadvantages. The use of group theory formalization allows the introduction of a structure to the otherwise unstructured set of equations produced by a set of constraints, and enforces the geometric underpinnings of the problem, making the algebraic geometry aspect more tractable. On the other hand, algebraic geometry responds to the question of geometric and topological inconsistencies, and to more complex constraint structures, removing the limit of trivial constraints treated using group theory alone.

[1] $SE(3)$, is the semi-direct product $R^3 \circ SO(3,R)$, where R^3 is the translational part, $SO(3,R)$ is the special orthogonal group, representing all right handed orthonormal 3-D frames and $\circ$ is the group multiplication operation.

2 Theoretical Background

In this section, the solution of the GCS/SF problem is discussed. In the first place, assuming that the problem can be expressed in terms of polynomial equations, the theoretical relevance of the algebraic geometry methods (Grobner Basis) will be discussed. On the other hand, given the high computational cost of Grobner Basis methods, alternative methods which produce smaller sets of variables for the problem at hand are needed. The direct connection between variables and physical degrees of freedom is a desirable characteristic of the set of variables chosen. In the past group-based variables have been employed in topology-only analysis [23], in this investigation they will provide a compact and physically meaningful set of variables for the GCS/SF problem.

2.1 Algebraic Geometry and Grobner Basis

In what follows, an introduction to an algebraic geometry technique called Grobner Basis construction will be attempted. Only those issues relevant to the GCS/SF problem are discussed. The interested reader is directed to [10, 5] for details. The following is some relevant terminology.

K[x1,x2,... xn]: ring of n-varied polynomials over the coefficient field K.

Ideal of F :

The Ideal generated by a set of polynomials $F = \{f_1, f_2, f_3, ...f_n\}$ is: $I_{K[x_1,x_2,...x_n]}\langle F\rangle = \{g_1 f_1 + g_2 f_2 + ... + g_n f_n | gi \in K[x_1, x_2, ...x_n]\}$. The notation is usually simplified to: $I\langle F\rangle$. One says that F is a *basis* for $I\langle F\rangle$.

Radical(F) : $\{f | \exists k \; s.t. \; f^k \in Ideal(F)\}$

Algebraic set V(I) : Given an ideal $I \in K[x_1, x_2, ...x_n]$, generated by the set $F = \{f_1, f_2, f_3, ...f_m\}$ its algebraic set $V(I)$ is defined by: $V(I) = \{x \in R^n | f(x) = 0, \forall f \in I\}$; therefore, $(f_i(x) = 0 \, \forall f_i \in F) \rightarrow (x \in V(I))$

Zero Dimension : An Ideal I is *zero dimensional* if $V(I)$ is finite.

Ordering : the set of variables $\{x_1, x_2, ..x_n\}$ is *totally ordered under the order* $\prec$ if $\forall x_i, x_j$, with $x_i \neq x_j$ either $x_i \prec x_j$ or $x_j \prec x_i$.

Lexicographic Order $\prec_l$: Given two terms $t_1 = x_1^{\alpha_1}.x_2^{\alpha_2}..x_n^{\alpha_n}$. and $t_2 = x_1^{\beta_1}.x_2^{\beta_2}..x_n^{\beta_n}$, then $t_1 \prec_l t_2$ iff $\exists i \leq n$ such that $\alpha_j = \beta_j$ for $i \prec j \preceq n$ and $\alpha_i < \beta_i$.

Degree : $deg(t) = deg(x_1^{\alpha_1}.x_2^{\alpha_2}..x_n^{\alpha_n}) = \alpha_1 + \alpha_2 + ... + \alpha_n$

Degree Order $\prec_d$: $t_1 \prec_d t_2$ iff $deg(t_1) < deg(t_2)$ or $deg(t_1) = deg(t_2)$ and $t_1 \prec_l t_2$

head(f), ldcf(f) : For a given order, and a given ring $K[x_1, x_2, .., x_n]$, $head(f)$ is the largest (in the sense of $\prec$) term in a polynomial f. $ldcf(f)$, the *leading coefficient* of f, is the coefficient of $head(f)$ in f. Therefore $f = ldcf(f).head(f) + tail(f)$.

Normal Form : Given $F = \{f_1, f_2, f_3, ...f_n\}$ and p where $F \subset K[x_1, x_2, ...x_n]$ and $p \in K[x_1, x_2, ...x_n]$, there exists a decomposition of p :
$p = NF(F,p) + \sum_{f_i \in F}(\alpha_{f_i} . f_i)$ $(\alpha_{f_i} \in K[x_1, x_2, ...x_n])$ such that $NF(F,p)$ cannot be further decomposed as $\sum_{f_i \in F}(\beta_{f_i} . f_i)$ with $\beta_{f_i} \in K[x_1, x_2, ...x_n]$. The term $NF(F,p)$ is called a *normal form of p with respect to F*. $NF(F,p)$ is a *residual* of the *reduction* of p *with respect to F*. The reduction process is denoted as $p \longrightarrow_F NF(F,p)$.

Grobner Basis : A Grobner Basis $GB \subset K[x_1, x_2, ...x_n]$ is a set of polynomials such that $NF(GB, f)$ for every f is unique; it does not depend on the *sequence* of reduction of f with respect to GB. Therefore, $f \longrightarrow_{GB} p_1$ and $f \longrightarrow_{GB} p_2$ imply $p_1 = p_2$. Also, if $NF(GB, f) = 0$ then $f \in I\langle GB \rangle$. If $NF(GB, f) \neq 0$ it implies some of the common roots of F are not roots of f; therefore the set of roots common to F and f is more restricted than the set of roots for F.

Reduced Grobner Basis : A Grobner Basis $GB = \{g_1, .., g_n\}$ is a *Reduced Grobner Basis* if:

1. for all $f_i \in GB$ $\;ldcf(f_i) = 1$
2. for all $f_i \in GB$ $\;NF(GB - \{f_i\}, f_i) = f_i$

Let $F = \{f_1, f_2, f_3, ...f_n\}$ be a set of polynomials, $F \subset K[x_1, x_2, ...x_n]$, and $I\langle F \rangle$ be its ideal. If another set $G = \{g_1, g_2, g_3, .., g_n\}$ is *basis* of $I\langle F \rangle$ then, every root of F is also root of G, and conversely.

Given a polynomial $f \in K[x_1, x_2, ...x_n]$ one may want to eliminate a term t of f with the help of another polynomial $g \in K[x_1, x_2, ...x_n]$ by multiplying the $head(g)$ by some term such that on subtracting the result from f, t disappears. For this to happen it is necessary that $g \prec f$. It is said then that *f is reduced with respect to g*. It is written as $f \xrightarrow{g} h$, where h is the result of the subtraction. In the process of iterated reductions with respect to elements of $K[x_1, x_2, .., x_n]$, the position of the h's in the ordering $\prec$ decays. One of two things may happen; f reduces to 0, or all the remaining g's are *bigger* than the final h, and therefore h (and f) cannot be further reduced. The last product of the reduction process is a *normal form of f with respect to* $K[x_1, x_2, ...x_n]$, $NF(K[x_1, x_2, ...x_n], f)$. In the described process, different sequences of reduction are possible, and they don't, in general, produce the same $NF(., f)$ result. If a set of polynomials F is used for the decomposition, $NF(F, f)$ can be considered as the part of f that cannot be expressed as a combination of the polynomials $f_i \in F$.

Several additional comments are pertinent at this point:

- Grobner Basis forces $NF(GB, F)$ to be unique, thus providing a way to examine whether an arbitrary polynomial p is in $I\langle F \rangle$ or not. If $p \in I\langle F \rangle$ then $NF(GB, p) = 0$. Otherwise, it represents an independent polynomial. Intuitively, Grobner Basis behaves in a manner analogous to a vector basis in linear spaces; if a vector can be expressed as a linear combination of the base vectors, it is in the space. Otherwise, a remainder term will represent the the component in the null space of the set of vectors.

- In a property described later (triangularity of elimination ideals), Grobner Basis presents a characteristic similar to triangulation of a matrix A in solving a linear system $A.x = b$. A triangular form allows the incremental determination of the solution point.

- In a Reduced Grobner Basis there is no redundancy in the polynomials present, since each polynomial is equal to its normal form with respect to the remaining ones. The value of this property in the solution of the system of polynomials is that it reduces to a minimum the polynomials to be manipulated and/or solved.

- An algorithm to calculate the Grobner Basis $GB(F)$ for the ideal generated by F, $I\langle F \rangle$, is provided by Buchberger in [5]. Several implementations are available in packages such as Mathematica, Maple, Macaulay, etc. The condition for termination of the Buchberger's algorithm relies heavily on the fact that a total order can be defined on the terms belonging to $K[x_1, x_2, ...x_n]$. Since a *decreasing* sequence (in the sense of $\prec$) of terms is finite, a reduction process of a polynomial p with respect to a set F is bound to stop [5, 10].

In next sections, the theoretical basis described here will be used to exploit the properties of Grobner Basis in the solution of the GCS/SF problem.

2.2 Algebraic Geometry and the GCS/SF Problem

Given a set of polynomials F (we assume $F = poly\_form(W, R)$ and also $S = feasible(W, R)$), it has an associated ideal $I\langle F \rangle$. For any set of polynomials F, the Grobner Basis GB(F) is an alternative set, which generates the same ideal $I\langle F \rangle$, but has important properties in characterizing the solution space and producing solution points.

The following are some of the properties of Grobner Basis:

1. $I\langle GB(F) \rangle = I\langle F \rangle$.
2. F is solvable iff $1 \notin$ GB(F).
3. Given a lexicographic order $x_1 \prec x_2 \prec ... \prec x_n$ $\forall i, s.t. 1 \leq i \leq n$, we have: $GB(F) \cap K[x_1, x_2, ...x_i]$ is a (reduced) Grobner Basis for the elimination Ideal $I_{K[x_1, x_2, ...x_n]}\langle F \rangle \cap K[x_1, x_2, ...x_i]$.

   This property establishes that GB(F) is triangular set; in the sense that GB(F) contains polynomials only in x_1, some others only in x_1, x_2, and so on, making the numerical solution a process similar to triangular elimination.

4. If G is the reduced Grobner Basis for an Ideal $I \subset K[x_1, x_2, ...x_n]$, I is *Zero Dimensional* iff $\forall x_i \in \{x_1, x_2, ..x_n\}$, G contains a polynomial whose head term is a pure power of x_i, i.e of the form x_i^d for some integer d.

   This property allows one to determine, by inspection, whether the set of polynomials has finitely or infinitely many solutions.

5. The Grobner basis $G1$ for a zero dimensional ideal I based on the order $\prec_m$ can be converted into another basis $G2$ under another ordering $\prec_l$

   This property allows one to compute *total degree* Grobner Bases for certain purposes, and only when it is required, to transform them into *lexicographic* Grobner Basis (computationally more expensive), provided that they correspond to a Zero Dimensional Ideal.

6. $\forall F, f : f \in Radical(F) \Leftrightarrow (1 \in GB(F \cup \{y.f - 1\}))$
   This property establishes that f presents the same zeros as F iff the system $F \cup \{y.f - 1\}$ is inconsistent, i.e. it is impossible for f not to be zero when F is.

These properties translate into propositions about the solvability and characteristics of the solution for the GCS/SF problem. Some of the consequences of the properties follow:

1. **Proposition 1**
   $S = solution(F)$ iff $S = solution(GB(F))$.
   This is a consequence of the fact that F and $GB(F)$ span the same polynomial ideal (Property 1). In the contex of the GCS/SF problem, a set of polynomials representing constraints is indirectly analyzed by calculating the Grobner Basis of its spanned ideal and solving it by using the properties discussed below.

2. **Proposition 2**
   $1 \in GB(F) \Rightarrow S = solution(F) = \Phi$
   Property 2 above establishes that if the field is algebraically closed, finding "1" or a constant polynomial in GB(F) implies the equation "0=1" leading to the fact that F has no solution in that field. However, the converse proposition has to be carefully explored; for our purposes (real solutions), a different property can be used:
   $1 \in GB(F) \rightarrow F$ has no solution. On the other hand, if $1 \notin GB(F)$, a solution exists, although it might be complex. Therefore, an additional check to ensure a *real* solution is needed.

3. **Proposition 3**
   If $I\langle F \rangle$ is a Zero-dimensional ideal, then the set F (and $GB(F)$) has a finite number of solutions. Therefore $S = feasible(W, R)$ has a finite number of configurations. The zero-dimensionality of I can be assessed by applying property 4 above.

4. **Proposition 4**
   Polynomial f is redundant to $F \Leftrightarrow (1 \in GB(F \cup \{y.f - 1\}))$ for a new variable y.
   Property 6 helps to determine whether an additional constraint is redundant by examining if the satisfaction of the new, additional constraint is unavoidable when the initial set of constraints is satisfied. An alternative test can be implemented by recalling that a polynomial f is redundant if its normal form $NF(GB(F), f) = NF(F, f)$ is equal to zero.

These properties and propositions provide a theoretical framework for the solution of the GCS/SF problem. The realization of these facts into an algorithm will be discussed in following sections.

2.3 An Algorithmic Solution to the GCS/SF Problem

This theoretical background can be summarized in the following macro-algorithm, in which the invariant clause for the loop is the existence of a set of non-redundant, consistent and multi-dimensional set of (constraint generated) polynomials. In the event of the addition of new constraints to the scene, the algorithm converts them into polynomial(s), and tests their redundancy (by using Proposition 4), inconsistency (Proposition 2) and Zero Dimensionality (Proposition 3). If the new constraint is redundant no action is taken; in the other two cases the invariant becomes false and the loop breaks. If the ideal has become Zero-dimensional a triangular Grobner Basis under some stated lexicographic order is extracted (Property 5) and solved (Property 3). Proposition 1 is the underlying basis of the algorithm, since it establishes that the $GB(F)$ faithfully represents F, with the same roots and ideal set. In the algorithm presented below, the propositions or properties relevant to some important instructions are displayed at the left hand side:

```
                {Pre: W a fixed scenario }
                    F = {}
                    GB_t = {}
                    do new relation R_i
                    {Inv:   F is consistent, non redundant,
                            Multi-dimensional }
                        R = R ∪ {R_i}
                        f = poly_form(W, R_i)
Proposition 2           if (1 ∈ GB_t(F ∪ {f})) then
                            stop ( system is inconsistent )
                        else
Proposition 4               if (f ∈ Radical(F)) then
                                skip ( f is redundant )
                            else
                                F = F ∪ {f}
Proposition 1                   GB_t = GrobnerBasis(F, ≺_t)
Proposition 3                   if ( ZeroDimension(GB_t) ) then
                                    break loop
                                else
                                    skip (next relation-constraint)
                                fi
                            fi
                        fi
                    od
Property 5          GB_l = GrobnerBasis(F, ≺_l)
Property 3          S = triangular_solution(GB_l)
                {Post:R = {Ri} a set of relations; S = feasible(W, R) }
```

The limitations of Grobner Basis, and for that matter, any symbolic algebraic geometry method solving this problem is the explosive computational complexity of the method, and its still unexplored behavior in dealing with real arithmetic. If F is a set in $K[x_1, x_2..x_n]$, with maximum exponent m, the Grobner Basis can contain polynomials of degree proportional to 2^{2^m} [10].

3 Methodology. Construction of Polynomial Set

Grobner Basis provides a series of properties that can be used to answer the questions about consistency, ambiguity, existence of solutions, dimension of the solution space, etc in the problem of Scene Feasibility. This section shows the methodologies developed to state the problem of scene feasibility in terms of polynomials from two points of view; canonical (based on group theory) and non-canonical forms. The differences between the two methods are the structure of the set, and the number and meaning of the variables involved.

The terms used are explained next; *entity* means *geometric entity*: point, line or plane. Each entity has an attached frame. Points are in the origin of their attached frame. Lines coincide with the X axis of their frame. Planes coincide with the Y-Z plane of their attached frame. The world W contains a set of topological (polyhedra and possibly non manifold objects) and geometrical (lines, planes, points) entities $S = \{e1, e2, ..en\}$. For the discussion at hand it is assumed that the entities are part of a body. F_{ij} is the known, fixed position of entity i with respect to the body frame j. R_k represent relations or constraints between entities. These relations are shown in Table 1. D_i represent displacements applied on the frames of the entities F_{ij}. M_k represent known, fixed relative positions among entities of a body. Notice that the M_k relative displacements can be expressed in function of the F_{ij} positions relative to the frame of body B_j.

Table 1: Constraint Relations and Polynomial Forms

relation	*entity 1*	*entity 2*	*vector equation*
P-ON-P	$p1$	$p2$	$p1 = p2$
P-ON-LN	$p1$	$LN = (p2, v2)$	$(p1 - p2) \times v2 = 0$
P-ON-PLN	$p1$	$PLN = (p2, n2)$	$(p1 - p2) \cdot n2 = 0$
LN-ON-LN	$LN = (p1, v1)$	$LN = (p2, v2)$	$v1 \times v2 = 0$ $(p1 - p2) \times v2 = 0$
LN-ON-PLN	$LN = (p1, v1)$	$PLN = (p2, n2)$	$(p1 - p2) \cdot n2 = 0$ $v1 \cdot n2 = 0$
PLN-ON-PLN	$PLN = (p1, n1)$	$PLN = (p2, n2)$	$(p1 - p2) \cdot n2 = 0$ $n1 \cdot n2 = \pm 1$

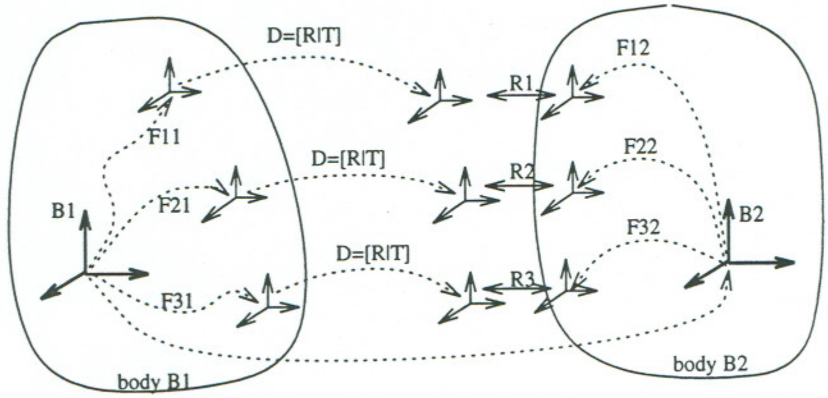

Figure 1: Methodology for Statement of Non-canonical form of Scene Feasibility

3.1 Methodology with Non Canonical Variables

Let be the disposition of entities as shown in Figure 1. Frames F_{i1} represent the position of distinguished element i ($i = 1..3$ in this case) of (and with respect to) body $B1$. Similar statements can be made about F_{i2} with respect to body $B2$. The goal is to find a position of $B1$ (assuming $B2$ is fixed) which satisfies the relations $F_{i1} - R_i - F_{i2}$. For example one may require that *point F11 be ON plane F12*. That means, $F_{11} - ON - F_{12}$.

The procedure for modeling the problem includes the following steps: (i) Assume a (unknown) displacement $D1 = \begin{bmatrix} Rot & T \\ 0 & 1 \end{bmatrix}$ with Rot a pure rotation and T a pure translation, which will place body $B1$ in the desired final position. (ii) Transform each entity to its new position; transformed vectors are $v_i' = Rot.v_i$; while transformed points are $p_i' = Rot.p_i + T$. (iii) Use Table 1 to model the proposed relations using the transformed entities [2]. (iv) Each relation (or constraint) produces a series of equations of the form $R_i(F_{i1}, D1, B2, F_{i2}) = 0$, which involves the corresponding entities F_{i1}, F_{i2}, the positions of the bodies $B1$ ($D1$) and $B2$, and the particular form of the relation R_i:

$$\begin{aligned} R_1(F_{11}, D1, B2, F_{12}) &= 0 \qquad (1) \\ R_2(F_{21}, D1, B2, F_{22}) &= 0 \\ R_3(F_{31}, D1, B2, F_{32}) &= 0 \end{aligned}$$

(v) Enforce the condition $det(Rot) = +1$ which imposes dexterous orthonormality to the matrix $Rot = [v_1\ v_2\ v_3]$. Orthonormality implies $\|v_i\| = 1, (i = 1..3)$; $v_j.v_i = 0, (i \neq j)$. Dexterity implies $v_1 \times v_2 = v_3$.

The procedure just described allows us to use the coordinates and parameters defining geometric entities as variables for the GCS/SF problem. An alternative method, using the parameters (called *canonical*) of the displacements of the entities in the scene is described ahead.

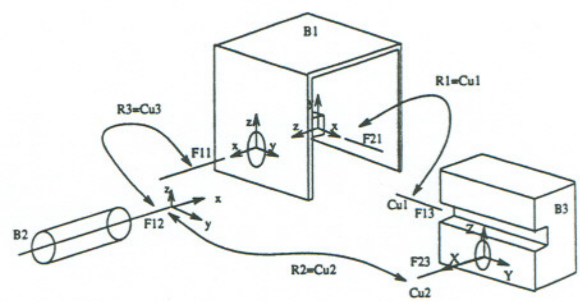

Figure 2: Three Body Assembly producing Non Trivial Constraints

3.1.1 Example 1. Non Canonical Variable Modeling.

This example highlights the differences in modeling and solution of the GCS/SF problem using canonical and non canonical variables. In this section we start with the non canonical modeling. Figure 2 (Adapted from [22]) shows the scenario being modeled. One may assume body $B1$ stationary and its frame coincident with the World Coordinate

[2] The proposed equations are not a minimal set. Some redundant equations are produced; for example $P - ON - LN$ can be expressed in two equations instead of three.

System. The following constraints are imposed on the entities:

- Line $F21 = (p_{21}, v_{21})$ is placed onto line $F13 = (p_{13}, v_{13})$ (Constraint R1)
- Line $F12 = (p_{12}, v_{12})$ is placed onto line $F11 = (p_{11}, v_{11})$ (Constraint R3)
- Line $F12 = (p_{12}, v_{12})$ is placed onto line $F23 = (p_{23}, v_{23})$ (Constraint R2)

Bodies $B2, B3$ are in (unknown) positions $D2$ and $D3$ respectively. The GCS/SF problem can be formulated following the steps above, using the vector equations of Table 1:

1. Expand the position matrices $D2$ and $D3$:

$$D2 = \begin{bmatrix} Rot_2 & T_2 \\ 0 & 1 \end{bmatrix}; \; D3 = \begin{bmatrix} Rot_3 & T_3 \\ 0 & 1 \end{bmatrix}$$

2. Transform the entities to the final position:

$$\begin{aligned} p'_{13} &= Rot_3.p_{13} + T_3; \quad v'_{13} = Rot_3.v_{13} \\ p'_{12} &= Rot_2.p_{12} + T_2; \quad v'_{12} = Rot_2.v_{12} \end{aligned} \tag{2}$$

3. Enforce the constraints:

$$\begin{aligned} v_{21} \times v'_{13} = 0; v_{21} \times (p_{21} - p'_{13}) = 0 \\ v'_{12} \times v_{11} = 0; v'_{12} \times (p'_{12} - p_{11}) = 0 \\ v'_{12} \times v'_{23} = 0; v'_{12} \times (p'_{12} - p'_{23}) = 0 \end{aligned} \tag{3}$$

4. Enforce orthonormality of Rot_2 and Rot_3.

5. Enforce dexterity of Rot_2 and Rot_3:

$$det(Rot_2) = +1; \quad det(Rot_3) = +1$$

This procedure produces a series of equations on the terms $D2 = \{D2_{i,j}\}$ and $D3 = \{D3_{i,j}\}$ whose solution space one wants to characterize.

3.2 Methodology with Canonical Variables

In this section an alternative representation of geometric constraints is introduced, which is based on mechanical constraints (joints), and their mathematical formalization, subgroups of the Euclidean group $SE(3)$. This modeling was first introduced by Herve in [9], and used by Angeles in [2, 3] and Torras et al in [21, 22] for topological analysis of mechanisms and constraints respectively. In this investigation, it will be used to provide a minimal and physically meaningful set of variables (called *canonical* by Herve) as an alternative to the non canonical set already discussed.

3.2.1 Subgroups of the $SE(3)$ Group

$SE(3)$ is the group of Euclidean displacements in 3D. $G1, G2$ and $G3$ represent displacements in $SE(3)$ and $\circ$ represents the composition of displacements. $SE(3)$ satisfies the group properties:

- Two displacements $G1, G2$ applied in sequence produce a new displacement $G3 = G1 \circ G2$.
- I is the null displacement in $SE(3)$. $G \circ I = I \circ G = G$
- For each $G \in SE(3)$ there is an inverse one G^{-1} which restores the affected entity to the original position $G \circ G^{-1} = G^{-1} \circ G = I$
- The effect of displacements is accumulative. If G's are applied in the order $G1, G2$ and $G3$, the following sequences are identical (associativity): $(G1 \circ G2) \circ G3 = G1 \circ (G2 \circ G3)$

$SE(3)$ presents subsets which are groups themselves, and which express certain common classes of displacements. They are called *subgroups*. For example, the subgroup of the rotations about a *given* axis u in the space, Ru, is a subset of $SE(3)$, and a group itself.

Given S_1, S_2 subgroups of $SE(3)$, S_1 is a **conjugate** of S_2 ($S_1 \sim S_2$) iff $\exists T \in SE(3)$ such that $S_1 = T^{-1} S_2 T$. The displacement T represents the *geometric* part of a particular constraint, while the canonical part contains the *topological* information; the number and types of degrees of freedom. The relation $A \sim B$ is an equivalence relation. It defines equivalence classes called *conjugacy classes* which represent the closure of a set of elements under the relation $S_1 \sim S_2$. Therefore the class can be represented by an element called *canonical* element. A desirable property for a canonical form is that it be minimal in the number of variables that specify it. Conjugacy classes have a canonical subgroup which represents any other subgroup in the class. A list of the conjugacy classes for the subgroups of $SE(3)$ and their canonical representation [9], as well as their degrees of freedom are shown in Table 2.

The concept of equivalence (conjugacy) allows us to name certain displacements in $SE(3)$ as "linear translations", "rotations", "planar slidings", etc, therefore making the link between subgroups of $SE(3)$ and kinematic *constraints*. For example, "rotations" are all transformations of the form

$$Ru(\theta) = B.\hat{Ru}.B^{-1} = B.twix(\theta).B^{-1}$$

with $B \in SE(3)$ (in the discussion that follows $twix(\theta)$ is a rotation about the X axis by θ, $XTOY$ means a rotation by 90^0 about the Z axis and $trans(x, y, z)$ indicates a general spatial translation). Therefore a rotation by θ degrees about the u axis in the space is a conjugate of any other by the same amount about another axis v in the space, because one can find a transformation B which transforms u into v. The displacement B represents the *geometric* part of a particular constraint, while the canonical part contains the *topological* information; the number and type of degrees of freedom. These degrees of freedom become the variables in the *canonical* modeling of the GCS/SF problem, as discussed in following sections.

A **constraint** between two entities is a set of relative displacements between them which, by definition, maintains invariant certain relations between the constrained entities. For example, a *planar sliding Gp* allows 2 translational and one rotational degree of freedom, while still ensuring planar contact between the two parts. A *rotational Ru* constraint preserves axial and radial relative distances, allowing one angular degree of freedom between the constrained entities.

The *GCS/SF* problem is stated as a series of constraints R_i relating F_{i1} with F_{i2} as shown in Figure 3, (corresponding to a two body system). The $R_i()$ constraints are in general composed by translations $T()$ and rotations $Rot()$, as dictated by Tables 3 and 2. Body $B1$ contains two features, whose frames are F_{11} and F_{21}. Corresponding features in body $B2$ are F_{12} and F_{22}. The goal is to find a final position of $B1$ (assuming $B2$ stationary), such that F_{11} relates to F_{12} and F_{21} relates to F_{22} satisfying the invariance dictated

Table 2: Conjugacy Classes and their Canonical Forms

Dof	*Symbol*	*Conjugacy Class*	*Canonical Subgroup*
1	Ru	Rotations about axis u	$\{twix(\theta)\}$
1	Tu	Translations along axis u	$\{trans(x,0,0)\}$
1	Hu,p	Screw movement along axis u, with pitch p	$\{trans(x,0,0).twix(px)\}$
2	Cu	Cylindrical movement along axis u	$\{trans(x,0,0).twix(\theta)\}$
2	Tp	Planar translation parallel to plane P	$\{trans(0,y,z)\}$
3	Gp	planar sliding along plane P	$\{trans(0,y,z).twix(\theta)\}$
3	So	Spherical rotation about center "o"	$\{twix(\psi).XTOY.twix(\phi).XTOY.twix(\theta)\}$
3	T	3D translation	$\{trans(x,y,z)\}$
3	Yv,p	Translating Screw axis v, pitch p	$\{trans(x,y,z).twix(px)\}$
4	Xv	3D translation followed by rotation about v	$\{trans(x,y,z).twix(\theta)\}$

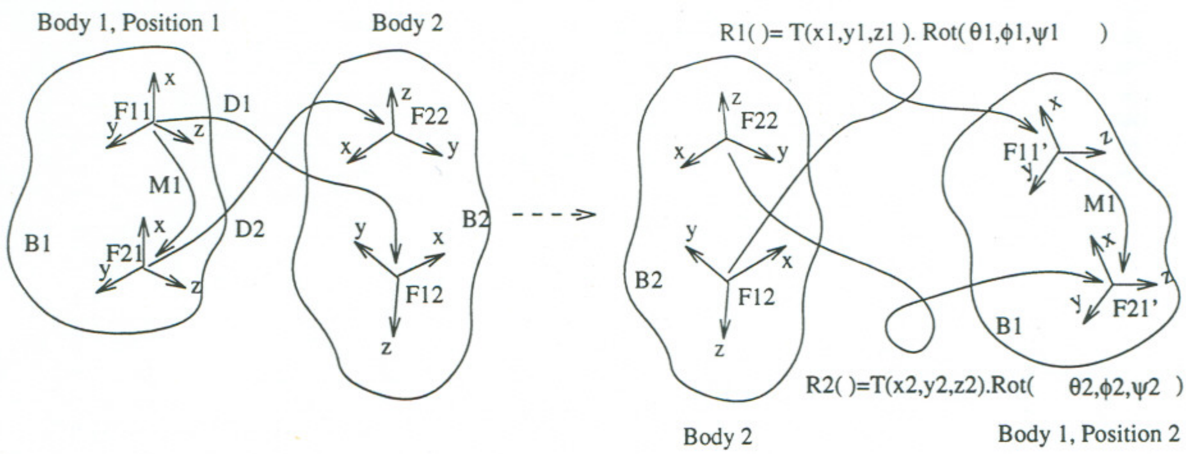

Figure 3: Two Body Example of Canonical Variable Modeling of the GCS/SF Problem

by $R_1()$ and $R_2()$ respectively. F'_{11} and F'_{21} denote frames F_{11} and F_{21} in the *final* configuration. The constraints $Ri()$ contain degrees of freedom one wants to instantiate so as to satisfy the required relations while, at the same time, enforcing the rigidity of the two bodies. One can establish matrix equations based on the relations and the rigidity conditions as follows:

1. Let $B1'$ be the final position of body $B1$.

2. The initial configuration of the World can be recorded by realizing that a relative displacement D_1 exists between feature F_{11} of body $B1$ and feature F_{12} in body $B2$, and D_2 from F_{21} in body $B1$ with respect to feature F_{22} in body $B2$.

$$B1.F_{11}.D_1 = B2.F_{12}; \quad B1.F_{21}.D_2 = B2.F_{22} \quad (4)$$

3. The features of the bodies in their final positions are forced to comply with the required constraints:

$$B1'.F_{11}.R_1() = B2.F_{12}; \quad B1'.F_{21}.R_2() = B2.F_{22} (5)$$
$$\rightarrow B1'.F_{11}.R_1().F_{12}^{-1} = B1'.F_{21}.R_2().F_{22}^{-1}$$

Notice that the degrees of freedom of $R_1()$ and $R_2()$ should get instanced in such a way the two movements $D_1.R_1$ and $D_2.R_2()$ comply with: (i) preserving the rigidity of the $B1$ and (ii) satisfying the desired constraint relations with $B2$.

4. The final position $B1'$ of body $B1$ is given by:

$$B1' = B2.F_{22}.R_1()^{-1}.F_{21}^{-1} \quad (6)$$

The above procedure can be generalized to the case in which there are several relations (constraints) $R_i()$ specified between two bodies or several relations among several bodies. Once the constraint equations are obtained by this procedure, the construction of the Grobner Basis and its interpretation are carried out in the manner described by the constraint management algorithm in section 2.3. The application of the above concepts is shown using an example in later sections.

3.2.2 Example 1. Canonical Variable Modeling.

For the Example corresponding to Figure 2 one notice that the relations $LN - ON - LN$ specified as constraints correspond to a the conjugacy of cylindrical displacements (see Table 3), and their canonical form can be expressed as $\hat{C_u} = trans(x,0,0).twix(\theta)$, (from Table 2). From the steps described above, one can determine that the matrix equations which express the constraint situation for this example are:

$$F21.Cu1(x_1,\theta_1).F13^{-1} = \quad (7)$$
$$F11.Cu3(x_3,\theta_3).Cu2(x_2,\theta_2).F23^{-1}$$

The matrix equations just presented represent the GCS/SF problem in polynomial terms using variables derived from the subgroups of $SE(3)$. In following sections the solution space for such a set of polynomials will be discussed.

4 Example 1. Solution Space for the GCS/SF Problem

In prior sections the two methodologies used to state the GCS/SF problem in terms of sets of polynomials were discussed. In this section a characterization of the solution space, which determines the set of feasible configurations is sought. As explained in sections before, algebraic geometry and in particular Grobner Basis provides a theoretical background to obtain such a characterization.

4.1 Grobner Basis with Non Canonical Variables

The following is the (lexicographic) reduced Grobner Basis for the set of polynomials based on the variables $D2_{ij}$ and $D3_{ij}$:

$$\begin{aligned}
\underline{D3_{34}} &= 0 \qquad (8)\\
\underline{D3_{33}} - 1 &= 0\\
\underline{D3_{32}} &= 0\\
\underline{D3_{31}} &= 0\\
\underline{D3_{24}^2} - 10\, D3_{24} &= 0\\
\underline{D3_{23}} &= 0\\
\underline{5\, D3_{22}} - 5 + D3_{24} &= 0\\
\underline{D3_{21}} &= 0\\
\underline{5\, D3_{14}} + D3_{24} &= 0\\
\underline{D3_{13}} &= 0\\
\underline{D3_{12}} &= 0\\
\underline{5\, D3_{11}} - 5 + D3_{24} &= 0\\
\underline{D2_{34}} - 2 &= 0\\
\underline{D2_{32}{}^2} + D2_{33}{}^2 - 1 &= 0\\
\underline{D2_{31}} &= 0\\
\underline{D2_{24}} - 5 &= 0\\
\underline{D2_{23}{}^2} + D2_{33}{}^2 - 1 &= 0\\
\underline{-D2_{22}} - D2_{23}\, D2_{32}\, D2_{33} + D2_{22}\, D2_{33}{}^2 &= 0\\
\underline{D2_{22}\, D2_{32}} + D2_{23}\, D2_{33} &= 0\\
\underline{D2_{22}\, D2_{23}} + D2_{32}\, D2_{33} &= 0\\
\underline{D2_{22}{}^2} - D2_{33}{}^2 &= 0\\
\underline{D2_{21}} &= 0\\
\underline{D2_{13}} &= 0\\
\underline{D2_{12}} &= 0\\
\underline{D2_{11}} + D2_{23}\, D2_{32} - D2_{22}\, D2_{33} &= 0
\end{aligned}$$

which is calculated based in the ordering: $D2_{11} \succ D2_{12} \succ D2_{13} \succ D2_{14} \succ D2_{21} \succ D2_{22} \succ D2_{23} \succ D2_{24} \succ D2_{31} \succ D2_{32} \succ D2_{33} \succ D2_{34} \succ D3_{11} \succ D3_{12} \succ D3_{13} \succ D3_{14} \succ D3_{21} \succ D3_{22} \succ D3_{23} \succ D3_{24} \succ D3_{31} \succ D3_{32} \succ D3_{33} \succ D3_{34}$ producing the following solution:

$$\begin{array}{ccc}
D2_{11} \to -1 & D2_{12} \to 0 & D2_{13} \to 0 \quad (9)\\
D2_{21} \to 0 & D2_{24} \to 5 & D2_{31} \to 0\\
D2_{34} \to 2 & D3_{11} \to -1 & D3_{12} \to 0\\
D3_{13} \to 0 & D3_{14} \to -2 & D3_{21} \to 0\\
D3_{22} \to -1 & D3_{23} \to 0 & D3_{31} \to 0\\
D3_{32} \to 0 & D3_{33} \to 1 & D3_{34} \to 0\\
D3_{24} \to 10 & D2_{23} \to -\sqrt{1 - D2_{33}{}^2} & \\
D2_{22} \to -D2_{33} & D2_{32} \to -\sqrt{1 - D2_{33}{}^2} &
\end{array}$$

The Grobner Basis (shown in Equation 8) is presented in triangular form, and the individual polynomials themselves have been arranged to have the *head*() term (under the order presented above and underlined in the equations) in the leftmost position. By examining the Grobner Basis one can detect that variables $D2_{14}$ and $D2_{33}$ are missing in the *head*() terms of polynomials. These two variables have very definite role in the $D2$ matrix (it is easy to see in this simple example); $D2_{14}$ represents a translational degree of freedom, while $D2_{33}$ represents a rotational degree of freedom about an unknown axis in the space, determined by the eigenvalues and eigenvectors of the submatrix $Rot_2 = \{D2_{ij}\}, (i = 1..3, j = 1..3)$ [4]. The solution implies (as expected) that body B3 is fixed while body B2 still has degrees of freedom left, represented in the variables $D2_{14}, D2_{33}$. Notice that in this case, non-instantiation of $D2_{33}$ immediately spreads to $D2_{32}, D2_{23}, D2_{22}$, since these values control the eigenvalues and eigenvectors of the matrix Rot_2. However this information is not self-evident from the solution set.

4.2 Grobner Basis with Canonical Variables

Once the equations 7 have been stated, the problem of finding their solution space is faced. In order to find the Grobner Basis of the Ideal generated by polynomials derived from equations 7, an ordering $x_1 \succ x_2 \succ x_3 \succ s\theta_1 \succ c\theta_1 \succ s\theta_2 \succ c\theta_2 \succ s\theta_3 \succ c\theta_3$ is defined, which produces a (lexicographic) Grobner Basis:

$$\begin{aligned}
\underline{s\theta_3{}^2} - 1 + c\theta_3{}^2 &= 0 \qquad (10)\\
\underline{c\theta_2} - c\theta_3 &= 0\\
\underline{s\theta_2} + s\theta_3 &= 0\\
\underline{c\theta_1} - 1 &= 0\\
\underline{s\theta_1} &= 0\\
\underline{x_2} + x_3 &= 0\\
\underline{x_1} &= 0
\end{aligned}$$

In this case, using canonical variables, immediately gives information on the degrees of freedom; for example, because of its position in the group equations, $c\theta_3$ represents $Cos(\theta_3)$, and from the Grobner Basis one knows that it is dependent on $c\theta_2$. Meanwhile, $x_1, c\theta_1, s\theta_1$ are completely instantiated, showing that the position of the body $B3$ is fixed. Body $B2$ is free to rotate about and translate along

Table 3: Entity Relations in form of Kinematic Joints

macro	*joint chain*	*kinematic joints in chain*	*dof*
P-ON-P	S_o	spherical	3
P-ON-LN	$T_1 \circ S$	linear translation, spherical	4
P-ON-PLN	$T_2 \circ S$	planar translation, spherical	5
LN-ON-LN	C_v	cylindrical	2
LN-ON-PLN	$G_P \circ R_w$	planar sliding, revolute	4
PLN-ON-PLN	G_P	planar sliding	3

axis $F12$. This is confirmed by the fact that x_2 and $s\theta_2$ do not pass the test for one-dimensionality of the Grobner Basis, which is in agreement with the fact that they are the variables which represent the remaining degrees of freedom. In this example again it is seen that the canonical variables present a convenient way to simplify the equations and to give geometric meaning to the polynomial solution process. Table 4 shows the comparative statistics between the two methods.

4.3 Example 2. Two Body System. Trivial Constraints

This example illustrates the intersection of two *trivial* constraints, corresponding to the conjugacy class of cylindrical C_v subgroups. Consider a scene in which there are two straight lines $LN_1 = (P_1, v_1)$ and $LN_2 = (P_2, v_2)$ (See Figure 4) expressed parametrically, and assumed to be rigidly linked to each other by a displacement M. Another set of lines, with similar conditions are given by $LN_3 = (P_3, v_3)$ and $LN_4 = (P_4, v_4)$. The proposed relations place LN_1 *ON* LN_3 and LN_2 *ON* LN_4, (being LN_3, LN_4 also rigidly joined) The goal of the problem is to find whether the relations can be satisfied, what displacement is to be performed on the rigid body holding LN_1 and LN_2 to achieve the goal, and the degrees of freedom that are afforded to the body holding LN_1 and LN_2 by the relationship. In this case, the remaining variable is a translational degree of freedom in the direction of the line axes. This problem was solved both by canonical and non canonical variables. The computer times for the solution are shown in Table 4.

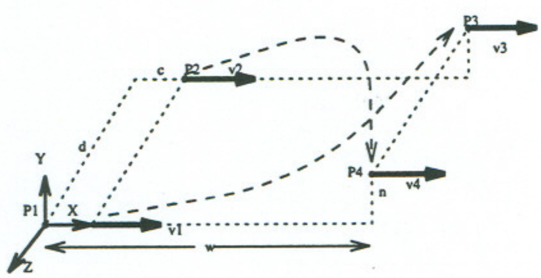

Figure 4: Simultaneous Line-to-Line Restriction between Pairs of Lines

4.4 Example 3. Four Body System, Non Trivial Constraints

This example (adapted from [22]) presents a more complex version of the non-trivial constraint system in Example 1. An additional body is introduced (a different body numbering than the one in Example 1 is used) according to Figure 5. The effect of this new entity is to increase the number of constraints, variables and equations one needs to manage. More importantly, the nature of the constraints is changed by this entity. The system no longer has a single loop of constraints but, instead, it has a number of different loops. As before, two methodologies are discussed; Non canonical vs. Canonical variables. Statistics comparing the execution times of the two methods are presented in Table 4.

The set of constraints imposed is the following:

Line $F21$ ON line $F13$ (Constraint R1)
Line $F31$ ON line $F12$ (Constraint R2)
Line $F22$ ON line $F23$ (Constraint R3)
Line $F14$ ON line $F32$ (Constraint R4)
Line $F14$ ON line $F33$ (Constraint R5)
Line $F14$ ON line $F11$ (Constraint R6)

Again, we assume body $B1$ is in the origin of the World, and bodies $B2, B3, B4$ are in positions $D2$,$D3$ and $D4$ respectively. Therefore $D2$, $D3$ and $D4$ are unknown rigid transformations matrices.

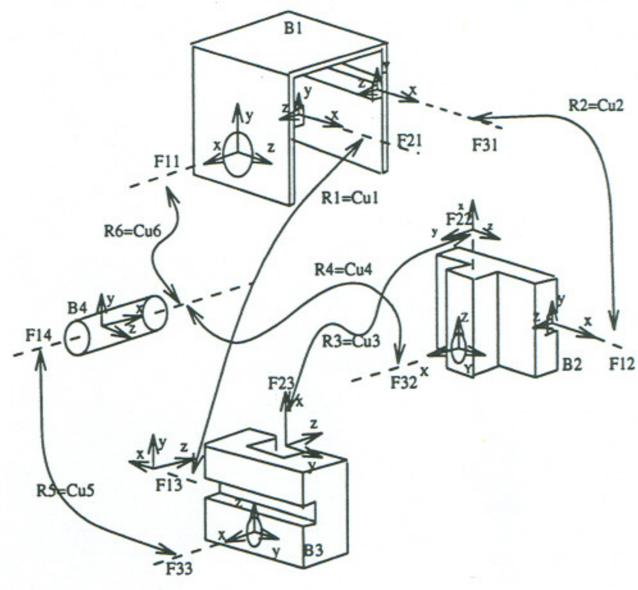

Figure 5: Four Body Assembly. Multi-Body, Non-trivial Constraint System

5 Conclusions

Grobner Basis of a set of polynomials $F = \{p1, p2, ..pn\}$ presents several properties for characterization of the algebraic variety of the ideal generated by F. If F is the polynomial expression of a GCS/SF problem ($F = poly\_form(W, R)$), these properties allow to determine the remaining degrees of freedom in the system, the redundancy of additional constraints, the (in)consistency of the F set,

Table 4: Statistics for Examples. Non Canonical vs Canonical Variables

Example	*Variable Type*	*variables*	*equations*	*GB size*	*time (secs)*
Example 1	Non canonical	20	30	24	6.08
Example 1	Canonical	9	15	7	0.51
Example 2	Non canonical	12	20	16	1.53
Example 2	Canonical	6	14	6	0.25
Example 3	Non canonical	36	54	37	39.01
Example 3	Canonical	18	42	16	2.16

etc. Grobner Basis presents, however, a high computational expense, which is lowered by the choice of a convenient set of (canonical) variables dictated by the conjugacy classes of the subgroups of the group $SE(3)$ of the Euclidean displacements. In many cases canonical variables present a compact representation, with direct physical meaning, therefore facilitating the interpretation of the results regarding the inventory of instantiated vs free variables. Also, Grobner basis allows one to deal with geometrical as well as topological data and inconsistencies, and is not limited to trivial constraints. Additional and future work ([18]) deals with the possibility of pre-processing the topological part of the constraint network by applying the method of Torras and Thomas [21, 23], and subsequent use of the Grobner basis method with a reduced set of constraints.

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