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Algebraic geometry and group theory in geometric constraint satisfaction for computer-aided design and assembly planning

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Mechanical design and assembly planning inherently involve geometric constraint satisfaction or scene feasibility (GCS/SF) problems. Such problems imply the satisfaction of proposed relations placed between undefined geometric entities in a given scenario. If the degrees of freedom remaining in the scene are compatible with the proposed relations or constraints, a set of entities is produced that populate the scenario satisfying the relations. Otherwise, a diagnostic of inconsistency of the problem is emitted. This problem appears in various forms in assembly planning (assembly model generation), process planning, constraint driven design, computer vision, etc. Previous attempts at solution using separate numerical, symbolic or procedural approaches suffer serious shortcomings in characterizing the solution space, in dealing simultaneously with geometric (dimensional) and topological (relational) inconsistencies, and in completely covering the possible physical variations of the problem. This investigation starts by formulating the problem as one of characterizing the solution space of a set of polynomials. By using theories developed in the area of algebraic geometry, properties of Grobner Bases are used to assess the consistency and ambiguity of the given problem and the dimension of its solution space. This method allows for the integration of geometric and topological reasoning. The high computational cost of Grobner Basis construction and the need for a compact and physically meaningful set of variables lead to the integration of known results on group theory. These results allow the characterization of geometric constraints in terms of the subgroups of the Special Group of Euclidean displacements in E^3, $SE(3)$. Several examples are developed which were solved with computer algebra systems (MAPLE and Mathematica). They are presented to illustrate the use of the Euclidean group-based variables, and to demonstrate the theoretical completeness of the algebraic geometry analysis over the domain of constraints expressible as polynomials.

1. Introduction and theoretical background

Spatial or geometry relations are of primary importance in computer-aided design, manufacture and process planning. As a result, spatial reasoning finds application in diverse areas such as assembly planning, computer vision, robotics, constraint-driven design and drafting, feature extraction, and machine tool selection and specification. We decompose the domain of spatial reasoning into two main areas; *static* and *dynamic* reasoning. Static reasoning problems are those concerned with fully and unambiguously defined entities. Typical problems include boolean queries testing a particular relation among entities, and construction queries that create entities satisfying relations with other given entities in the world. These problems have well defined (though not unique) answers. A sound taxonomy of the problems in flat entity worlds was compiled and developed in Shamos (1978) and Shamos and Preparata (1985). In this taxonomy, the problems are divided into convex hulls, inclusion, intersection, and closest point problems. Non-flat entities, on the other hand, engender a new broad (and difficult) set of problems, in which lines and surfaces are curved, and the computational expense of even the simplest operations grows very rapidly when compared with their counterparts in the domain of flat entities. In dynamic reasoning, a set of geometric entities is specified by geometric relations (also called *constraints*) in the context of a given world. As the set of relations varies, the position (and the very existence) of the entities in the space changes. For example, a specification might require the creation of a line perpendicular to another line. With no other conditions, the result is an infinite complex of possible answers. In other cases the relation specified between objects might be internally inconsistent making their creation or modification impossible. For example, an inconsistent specification might request a line being simultaneously perpendicular to two non-parallel planes, thereby producing an empty solution space. Producing a configuration of geometric entities satisfying a set of spatial relations is called scene feasibility (SF) or geometric constraint satisfaction (GCS) problem.

1.1. *Problem statement*

The GCS/SF problem in a flat entity world can be stated as follows: Let a World W be characterized by a

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coordinate reference frame in E^3, and S be a set of geometric entities $S = \{e_1, \ldots, e_n\}$ (points, straight lines, planes). A set of spatial relations among pairs of entities $R = \{R_{i,j,k}\}$ are specified, where $R_{i,j,k}$ is the kth relation between entities i and j. The goal is to find the position of each entity e_i in the world W, consistent with all relations R specified on it, $R_{i,W}$. The world W contains a fixed scenario with some basic, well defined entities in which the new set of entities S satisfying R will eventually be instanced.

If S satisfies the relations in R in the context of the basic world W, it is said that *S is feasible for W and R*. It is written as $S = feasible(W, R)$. This problem can be translated into one of characterizing the solution space of a set of polynomial equations $F = \{f_1, f_2, \ldots, f_n\}$; therefore *$F$ is the polynomial form of the problem* (W, R); and is written as $F = poly\_form(W, R)$. Since *S is a solution to F*, it is written as $S = solution(F)$.

The research addressing the GCS/SF problem has taken two main directions: (i) the mathematical approach, which studies the theoretical foundations of the problem, and (ii) the procedural approach, in which the process of construction of the scene is simulated by a constraint management system.

In the procedural approach the fundamental tasks of the system are to maintain the information about the rotational and translational degrees of freedom and to simplify complex systems of constraints (Turner *et al.*, 1992; Kramer, 1992). An advantage of the procedural approaches is the practicality of the solution strategy, which simulates the process of scene construction that a human would follow in the constraint satisfaction. These approaches, however, are based on simplifications such as the topological-only analysis of the structure of the problem, or the assumption of separability of rotational and translational degrees of freedom. As discussed later, one of the most difficult characteristics of the GCS/SF problem is the joint influence of geometry and topology on the structure of the solution space. Ignoring this dependency, for example, leads to well-known inconsistencies in applying the Kutzbach–Grubler criterion in the domain of mobility analysis (Kramer, 1992). Other limitations of procedural approaches are the need for algorithms customized for each application (Kramer, 1992) and the incorrect assessment of the remaining degrees of freedom in cases where no decoupling of rotational–translational degrees of freedom is possible (Turner *et al.*, 1992). Owing to the above limitations, it is the goal of this investigation to explore the theoretical (algebraic) formalization of the GCS/SF problem. The algebraic approach, discussed next, does not present theoretical limitations of domain, number of entities, or types of constraints. It is expected that future combined efforts (algebraic and procedural) will effectively address the GCS/SF problem.

For an algebraic formalization, it will be shown that finding a solution for the GCS/SF problem is equivalent to finding a common root to the set of polynomial equations that express the constraints involved. Therefore methodologies for finding solutions to sets of polynomial equations are relevant in this investigation. A characterization of the solution space, rather than a particular solution, is required. Important issues in this characterization are the finiteness of the solution space (i.e., the existence of discrete configurations in which the entities satisfy the constraints), the dimension of the solution complex (i.e., the degrees of freedom of the entities or the rigid body motions that are allowed under a specified set of relationships), and the detection of redundant relations.

A numerical approach to the solution of a polynomial set of equations for geometric purposes has been investigated in Celaya and Torras (1990), Rocheleau and Lee (1987) and Ambler and Popplestone (1975). Numerical solutions, besides their problems with convergence, only sample the solution space to produce a single solution. Often in design and planning environments, the truly interesting information includes the dimension of the solution space, which is related to the remaining degrees of freedom, the consistency and/or redundancy of the constraint set, etc. Therefore, even in the event of perfect convergence, numerical solutions provide only a partial answer to the problem.

A full solution to the GCS/SF problem requires the determination of the algebraic set of the *ideal* of the corresponding polynomial set (i.e., the set of common roots of the polynomials). Algebraic geometry allows the symbolic characterization of the algebraic set (multiplicity of solutions, dimension of solution space, etc). It is, from the theoretical point of view, a well defined problem, whose solution can be constructed by several techniques in algebraic geometry such as the construction of the Grobner Basis (Buchberger, 1989; Hoffmann, 1989, Mundy *et al.*, 1991) and methods of resultants (Kapur and Lakshman, 1992). A related subproblem, automatic geometric theorem proving, can be also solved by the Characteristic Sets method (Chou, 1987, 1990). This paper will not explore this approach because it requires the initial generation of a hypothesis. This would be an inherent disadvantage in the context of the GCS/SF problem.

In this investigation it will be shown that the advantages of algebraic geometry applied to this problem are: (i) the complete characterization of the solution, (ii) the easy analysis of incrementally added constraints, and (iii) the theoretical robustness of the techniques.The drawback common to algebraic geometry techniques is the large computational expense of calculating the basis for the ideal of a polynomial set (Kapur and Lakshman, 1992). On the understanding that these techniques are general and independent of the domain being modeled, it is possible to offset their disadvantages by exploiting characteristics particular to the modeled domain; geometry, in this case. The next paragraph elaborates on this notion.

While algebraic geometry provides one approach to this problem, group theory and in particular Euclidean Groups (Angeles, 1982, 1988; Herve, 1978) has been used to reason about constraints placed on mechanical objects. The kinematic constraints can be expressed in terms of contact between (flat) geometric entities such as points, lines and planes. By expressing the constraints in terms of the subgroups of the Special Euclidean Group of displacements in E^3, $SE(3)$ ($SE(3)$ is actually the semi-direct product $R^3 \times SO(3, R)$, where R^3 is the translational part and $SO(3, R)$, the special orthogonal group, represents all oriented orthonormal 3-frames), the simultaneous enforcement of constraints can be expressed as the intersection of the sets conforming the corresponding subgroups, and the composition of constraints can be characterized as the product of the corresponding sets using the group operation (Angeles, 1982, 1988; Herve, 1978; Ledermann, 1953). This method presents the advantage of expressing the problem in terms of physically meaningful and independent variables, thereby stressing the geometric origin of the problem. It is limited, however, in the sense that it does not deal with geometric inconsistencies, and it considers single (also called *trivial* (Angeles, 1988; Herve, 1978)) constraints, owing to the fact that composition of subgroups is not a closed operation, i.e., the composition of two subgroups is not a subgroup in general (Celaya and Torras, 1990; Thomas and Torras, 1988, 1989).

This investigation (partly introduced by the authors in Ruiz and Ferreira (1994) and Ruiz *et al.* (1994)) highlights the mapping between properties of polynomial ideals and their Grobner Bases and the GCS/SF problem. Although the properties of Grobner Bases have been studied in the context of general polynomial fields, their application in the GCS/SF problem has not been formalized. As a result of this mapping, a constraint management algorithm is proposed and applied. Next, we introduce the integration of the algebraic geometry-based approach with the formalisms provided by a group-theoretic analysis of the constraint set. By doing so, the two approaches complement each other to reduce the effects of their individual disadvantages. The use of a group-theoretic formulation of the constraints introduces structure and physical meaning into an otherwise unstructured set of equations and, by doing so, has the potential of making the construction of the Grobner Basis more efficient, with its results easier to interpret as degrees of freedom in a given scenario. Viewed the other way, the Grobner Basis construction can be seen as a mechanical procedure to replace the reduction based on group intersection. Further, it simultaneously evaluates topological and geometrical consistency.

In this paper, Section 2 presents a formal procedure that states GCS/SF in terms of sets of polynomials. Section 3 discusses the mapping between GCS/SF and the properties of Grobner Basis. Section 4 sets group-theoretical foundations and applies them to the modeling of GCS/SF. Section 5 introduces the group-based modeling methodology and develops examples comparing the two methodologies explored. Section 6 discusses the results of the examples and Section 7 draws the general conclusions of the investigation and outlines the issues to be addressed in future research.

2. Methodology with non-canonical variables

The following methodology for expressing the GCS/SF problem uses as variables the (unknown) elements of the position matrix of an entity in the space. In this work, such variables are called *non-canonical* variables. A canonical set is discussed later.

As formulated here, the GCS/SF problem considers *contact* constraints (Table 1, column 1) between flat geometrical entities (points, lines, and planes). The constraints can be expressed as sequences of lower kinematic pairs (joints that make surface contact between bodies) that are commonly used in engineering design. They can also be written as sets of polynomials in terms of the variables that characterize the entities (Table 1, column 4).

2.1 Construction of polynomial set

The following conventions will be used. (i) *Entity* means geometric entity: point, line or plane. Each entity has an attached frame. Points are in the origin of their attached frame. Lines coincide with the X axis of their frame. Planes coincide with the Y–Z plane of their attached

Table 1. Elementary relations and polynomial forms

Relation	*Entity 1*	*Entity 2*	*Vector equation*
$P-ON-P$	p_1	p_2	$p_1 = p_2$
$P-ON-LN$	p_1	$LN = (p_2, v_2)$	$(p_1 - p_2) \times v_2 = 0$
$P-ON-PLN$	p_1	$PLN = (p_2, n_2)$	$(p_1 - p_2) \cdot n_2 = 0$
$LN-ON-LN$	$LN = (p_1, v_1)$	$LN = (p_2, v_2)$	$v_1 \times v_2 = 0; (p_1 - p_2) \times v_2 = 0$
$LN-ON-PLN$	$LN = (p_1, v_1)$	$PLN = (p_2, n_2)$	$(p_1 - p_2) \cdot n_2 = 0; v_1 \cdot n_2 = 0$
$PLN-ON-PLN$	$PLN = (p_1, n_1)$	$PLN = (p_2, n_2)$	$(p_1 - p_2) \cdot n_2 = 0; n_1 \cdot n_2 = \pm 1$

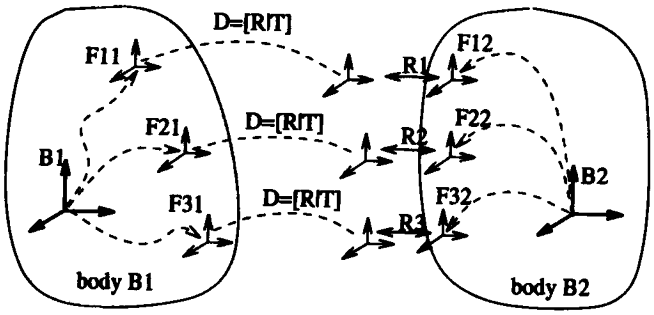

Fig. 1. Methodology for statement of non-canonical form of scene feasibility problem.

frame. (ii) F_{ij} is the known, fixed relative position of feature e_i inside body (frame) B_j. (iii) R_i represent relations or constraints (Table 1, column 1) between entities. (iv) D_i represent displacements applied on the frames of the features F_{ij}.

Let the disposition of entities be as shown in Fig. 1, where B_2 can be assumed stationary with no loss of generality. The goal is to find a position of B_1 that satisfies the relations $F_{i1} - R_i - F_{i2}$ ($i = 1 \ldots 3$ in this case). For example one might require that *point* F_{11} *(be) ON plane* F_{12} ($F_{11} - ON - F_{12}$).

The procedure for modeling the problem in terms of sets of polynomials is:

1. Assume a (unknown) displacement

$$D = \begin{bmatrix} Rot & T \\ 0 & 1 \end{bmatrix}$$

which will place body B_1 in the desired final position.

2. Transform each entity to its new position: $D.F_{ij}$.

3. Use Table 1 to model the proposed relations using the transformed entities. Notice that the proposed equations are not a minimal set; some redundant equations are produced; for example $P - ON - LN$ can be expressed in two equations instead of three.

4. Each relation (or constraint) produces a set of polynomials of the form $R_i(F_{i1}, D, B_2, F_{i2}) = 0$, which involves the corresponding entities F_{i1}, F_{i2}, the positions of the bodies D, B_2, and the particular form of the relation R_i:

$$R_1(F_{11}, D, B_2, F_{12}) = 0; R_2(F_{21}, D, B_2, F_{22}) = 0;$$
$$R_3(F_{31}, D, B_2, F_{32}) = 0 \qquad (1)$$

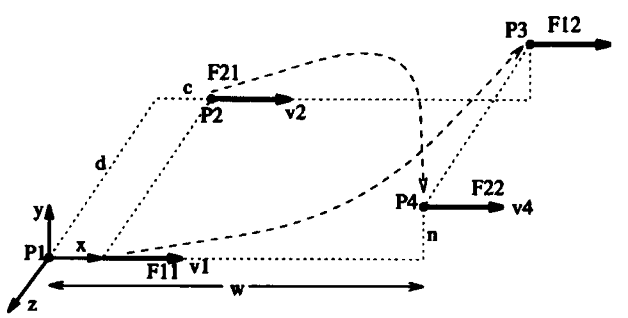

Fig. 2. Simultaneous line-to-line restriction between pairs of lines.

2.2. *Double-peg-hole example: non-canonical modeling*

Consider a scene in which there are two straight lines $LN_1 = (P_1, v_1)$ and $LN_2 = (P_2, v_2)$ (See Fig. 2) expressed parametrically, and assumed to be rigidly linked to each other by a displacement M. Another set of lines, with similar conditions, are given by $LN_3 = (P_3, v_3)$ and $LN_4 = (P_4, v_4)$. The proposed relations place LN_1 *ON* LN_3 and LN_2 *ON* LN_4, (being LN_3 and LN_4 also rigidly joined).The goal is to find out whether the relations can be satisfied, what displacement is to be performed on the rigid body $LN_1 - LN_2$ to achieve the goal, and the degrees of freedom that are afforded to the body $LN_1 - LN_2$ by the relationship. The physical interpretation of this GCS/SF problem is a situation in which, for example, two pegs in a body have to be simultaneously inserted into two holes in a wall. The problem can be stated as follows:

1. Apply a (still unknown) rigid displacement D to LN_1 and LN_2. D is formed by a rotation Rot and a translation T:

$$Rot = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}; T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}. \qquad (2)$$

The transformed entities are

$$P_1' = T + Rot.P_1; v_1' = Rot.v_1; P_2' = T + Rot.P_2;$$
$$v_2' = Rot.v_2. \qquad (3)$$

2. The specified relations impose the following conditions (expressed in vector terms for simplicity):

$$\left.\begin{array}{l} (P_1' - P_3) \times v_3 = 0 \ \ (P_1' \in LN_3); v_1' \times v_3 = 0 (v_1' \parallel v_3); \\ (P_2' - P_4) \times v_4 = 0 \ \ (P_2' \in LN_4); v_2' \times v_4 = 0 (v_4' \parallel v_4); \\ \quad \det(Rot) = +1; \end{array}\right\} (4)$$

The condition $\det(Rot) = +1$ imposes dexterous orthonormality to the matrix $Rot = [v_1 v_2 v_3]$. Orthonormality implies $\|v_i\| = 1, (i = 1..3)$; $v_j.v_i = 0, (i \neq j)$. Dexterity implies $v_1 \times v_2 = v_3$. The corresponding polynomials are presented and discussed in later sections (Eqn (5)).

The equations arrived at are polynomials, whose solution determine the matrix D, and therefore the position of the (frame of) body B_1. Following sections explore the techniques for characterizing the common solutions for such a polynomial set.

3. Grobner Basis and the GCS/SF problem

In what follows, an introduction to an algebraic geometry technique called Grobner Basis construction will be attempted. Only those issues relevant to the GCS/SF problem are discussed. The interested reader is directed to Buchberger (1989) and Hoffmann (1989) for details. The following is some relevant terminology.

$K[x_1, x_2, \ldots, x_n]$: ring of n-varied polynomials over the coefficient field K.

- *Algebraic closure*: the algebraic closure of a field K, $\overline{K}$, is the field of all roots of all polynomials in $\underline{K}[x_1, x_2, \ldots, x_n]$. If K is R, the set of real numbers, then $\overline{K}$ is the set of complex numbers, C.
- *Ideal of F*: the Ideal of a polynomial set $F = \{f_1, f_2, f_3, \ldots, f_n\}$ is

$$I_{K[x_1,x_2,\ldots,x_n]}\langle F\rangle = \{g_1.f_1 + g_2.f_2 + \ldots + g_n.f_n | g_i \in K[x_1, x_2, \ldots, x_n]\}.$$

The notation is usually simplified to: $I\langle F\rangle$. It is said that F is a *basis* for $I\langle F\rangle$.

- *Radical(F)*: $\{f | \exists\, k \text{ s.t. } f^k \in I\langle F\rangle\}$.
- *Algebraic set V(I)*: given an Ideal $I \in K[x_1, x_2, \ldots, x_n]$, generated by the set $F = \{f_1, f_2, \ldots, f_m\}$, its algebraic set $V(I)$ is defined by:

$$V(I) = \{x \in \overline{K}^n | f(x) = 0, \forall f \in I\};$$

therefore, $(f_i(x) = 0\ \forall f_i \in F) \rightarrow (x \in V(I))$.

- *Zero dimension*: an Ideal I is *zero dimensional* if $V(I)$ is finite.
- *Ordering*: the set of variables $\{x_1, x_2, \ldots, x_n\}$ is *totally ordered under the order* $\prec$ if

$$\forall\, x_i, \forall\, x_j, x_i \neq x_j \longrightarrow x_i \prec x_j \text{ or } x_j \prec x_i.$$

- *Lexicographic order* $\prec_l$: given two terms $t_1 = x_1^{\alpha_1}.x_2^{\alpha_2}\ldots x_n^{\alpha_n}$ and $t_2 = x_1^{\beta_1}.x_2^{\beta_2}\ldots x_n^{\beta_n}$, then $t_1 \prec_l t_2$ iff $\exists i \leq n$ such that $\alpha_j = \beta_j$ for $i < j \leq n$ and $\alpha_i < \beta_i$.
- *Degree*: $deg(t) = deg(x_1^{\alpha_1}.x_2^{\alpha_2}\ldots x_n^{\alpha_n}) = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.
- *Degree order* $\prec_d$: $t_1 \prec_d t_2$ iff $deg(t_1) < deg(t_2)$ or $deg(t_1) = deg(t_2)$ and $t_1 \prec_l t_2$.
- *head(f), ldcf(f)*: for a given order, and a given ring $K[x_1, x_2, \ldots, x_n]$, *head(f)* is the largest (in the sense of $\prec$) term in f. *ldcf(f)*, the *leading coefficient* of f, is the coefficient of *head(f)* in f. Therefore $f = ldcf(f).head(f) + tail(f)$.
- *Normal form*: given $F = \{f_1, f_2, \ldots, f_n\}$, $F \subset K[x_1, x_2, \ldots, x_n]$, and $p \in K[x_1, x_2, \ldots, x_n]$, there exists a decomposition of p: $p = NF(F,p) + \sum_{f_i \in F}(\alpha_{f_i}.f_i)$ (with $\alpha_{f_i} \in K[x_1, x_2, \ldots, x_n]$) such that $NF(F,p)$ cannot be further decomposed as $\sum_{f_i \in F}(\beta_{f_i}.f_i)$ with $\beta_{f_i} \in K[x_1, x_2, \ldots, x_n]$. The term $NF(F,p)$ is called a *normal form of p with respect to F*; it is a *residual* of the *reduction* of p *with respect to F*. The reduction process is denoted as $p \longrightarrow_F NF(F,p)$.
- *Grobner Basis*: a Grobner Basis $GB \subset K[x_1, x_2, \ldots, x_n]$ is a set of polynomials such that $NF(GB,f)$ for every f is unique; it does not depend on the *sequence* of reduction of f with respect to GB. Therefore, $f \longrightarrow_{GB} p_1$ and $f \longrightarrow_{GB} p_2$ imply $p_1 = p_2$ (the converse is not true). Also, if $NF(GB,f) = 0$ then $f \in I\langle GB\rangle$. If $NF(GB,f) \neq 0$ it implies that the set of roots common to F and f is more restricted than the set of roots to F.
- *Reduced Grobner Basis*: a Grobner Basis $GB = \{g_1, \ldots, g_n\}$ is a *Reduced Grobner Basis* if:
  1. $\forall f_i \in GB \quad ldcf(f_i) = 1$;
  2. $\forall f_i \in GB \quad NF(GB - \{f_i\}, f_i) = f_i$.

Let $F = \{f_1, f_2, f_3, \ldots, f_n\}$ be a polynomial set in $K[x_1, x_2, \ldots, x_n]$, and $I\langle F\rangle$ be its ideal. If another set $G = \{g_1, g_2, g_3, \ldots, g_n\}$ is *basis* of $I\langle F\rangle$ then every root of F is also root of G, and vice versa.

Given a polynomial $f \in K[x_1, x_2, \ldots, x_n]$, a term t of f can be eliminated with the help of another polynomial $g \in K[x_1, x_2, \ldots, x_n]$ by multiplying the *head(g)* by some term such that on subtracting the result from f, t disappears. For this to happen it is necessary that $g \prec f$. It is said then that *f is reduced with respect to g*. It is written as $f \xrightarrow{g} h$, where h is the result of the subtraction. In the process of iterated reductions with respect to elements of $K[x_1, x_2, \ldots, x_n]$, the position of the h's in the ordering $\prec$ decays. One of two things may happen: f reduces to 0, or all the remaining g's are *bigger* than the final h, and therefore f cannot be further reduced. The last product of the reduction process is a *normal form of f with respect to* $K[x_1, x_2, \ldots, x_n]$, $NF(K[x_1, x_2, \ldots, x_n], f)$. In the described process, different sequences of reduction are possible, and in general they do not produce the same $NF(.,f)$ result. If a set of polynomials F is used for the decomposition, $NF(F,f)$ can be considered as the part of f that cannot be expressed as a combination of the polynomials $f_i \in F$.

Several additional comments are pertinent at this point:

- Grobner Basis forces $NF(GB, F)$ to be unique, thus providing a way to examine whether an arbitrary polynomial p is contained in $I\langle F\rangle$or not. If $p \in I\langle F\rangle$ then $NF(GB, p) = 0$. Otherwise, it represents an independent polynomial.
- In a Reduced Grobner Basis there is no redundancy in the polynomials present, because each polynomial is equal to its normal form with respect to the remaining ones.
- An algorithm to calculate the Grobner Basis $GB(F)$ for a polynomial Ideal $I\langle F\rangle$ is provided by Buchberger (1989). Several implementations are available in packages such as Mathematica, Maple, and Macaulay. The condition for termination of the Buchberger algorithm relies on the fact that a total order can be defined on the terms belonging to $K[x_1, x_2, \ldots, x_n]$ (Buchberger, 1989; Hoffmann, 1989). Since a *decreas-*

ing sequence (in the sense of $\prec$) of terms is finite, a reduction process of a polynomial p with respect to a set F is bound to stop. Details of the Buchberger algorithm are discussed in Buchberger (1989) and Hoffmann (1989).

In the next sections, the theoretical basis developed here will be used to exploit the properties of Grobner Basis in the solution of the GCS/SF problem.

3.1. *Algebraic geometry and the GCS/SF problem*

Given a set of polynomials F (it is assumed that $F = poly\_form(W, R)$ and also $S = feasible(W, R)$), it has an associated Ideal $I\langle F\rangle$. For any polynomial set F, the Grobner Basis $GB(F)$ is an alternative set, which generates the same Ideal $I\langle F\rangle$, but has important properties in characterizing the solution space and producing feasible solutions:

1. $I\langle GB(F)\rangle = I\langle F\rangle$.
2. F is solvable in $\overline{K}$ (the algebraic closure of K) iff $1 \notin GB(F)$.
3. Given a lexicographic order $x_1 \prec x_2 \prec, \ldots, \prec x_n$ $\forall\, i$, s.t. $1 \leq i \leq n$: $GB(F) \cap K[x_1, x_2, \ldots, x_i]$ is a (reduced) Grobner Basis for the elimination Ideal $I_{K[x_1,x_2,\ldots,x_n]}\langle F\rangle \cap K[x_1, x_2, \ldots, x_i]$. $GB(F)$ is triangular set: $GB(F)$ contains polynomials only in x_1, some others only in x_1, x_2, etc. Solutions for x_1 are used to find x_2; $\{x_1, x_2\}$ are used to find x_3, and so on.
4. If G is the Reduced Grobner Basis for an Ideal $I \in K[x_1, x_2, \ldots, x_n], I$ is zero dimensional iff $\forall\, x_i \in \{x_1, x_2, \ldots, x_n\}$, G contains a polynomial whose head term is a pure power of x_i, i.e. of the form x_i^d for some integer d. This property allows the determination, by inspection, whether the set of polynomials has finitely or infinitely many solutions.
5. The Grobner basis G_1 for a zero-dimensional Ideal I based on the order $\prec_m$ can be converted into another basis G_2 under another ordering $\prec_l$. This property allows the computation of total degree Grobner Bases for certain purposes, and, only when it is required, their transformation into lexicographic Grobner Bases (computationally more expensive), provided that they correspond to a zero-dimensional Ideal.
6. $\forall\, F, f : f \in Radical(F) \Leftrightarrow (1 \in GB(F \cup \{y.f - 1\}))$ This property establishes that f presents the same zeros as F iff the system $F \cup \{y.f - 1\}$ (with $y \notin \{x_1, x_2, \ldots, x_n\}$), is inconsistent, i.e. it is impossible for f not to be zero when F is.

Some of the consequences of the properties in the GCS/SF context follow:

Proposition 1.
$S = solution(F)$ iff $S = solution(GB(F))$.
As a consequence of Property 1, in the context of the GCS/SF problem, $F = poly\_form(W, R)$ can be analyzed by calculating $GB(F)$ and solving it by using the properties discussed below.

Proposition 2.
$1 \in GB(F) \Rightarrow S = solution(F) = \Phi$.
Property 2 above establishes that $1 \in GB(F)$ is a necessary and sufficient condition for an empty solution space in $\overline{K}^n$. In contrast, if $1 \notin GB(F)$, a solution space exists in $\overline{K}^n$. A feasible S requires an additional check to ensure that $S \in R^n$.

Proposition 3.
If $I\langle F\rangle$ is a zero-dimensional Ideal, then the set F (and $GB(F)$) has a finite number of solutions. Therefore $S = feasible(W, R)$ has a finite number of configurations. The zero-dimensionality of I can be assessed by applying property 4 above.

Proposition 4.
Polynomial f is redundant with respect to $F \Leftrightarrow (1 \in GB(F \cup \{y.f - 1\}))$ for a new variable y.
Property 6 helps to determine whether an additional constraint is redundant by examining whether the satisfaction of the new, additional constraint is implied when the initial set of constraints is satisfied. An alternative test can be implemented by recalling that a polynomial f is redundant if its normal form $NF(GB(F), f) = NF(F, f)$ is equal to zero.

These properties and propositions provide a theoretical framework for the solution of the GCS/SF problem. The realization of these facts into an algorithm will be discussed in the following sections.

3.2. *An algorithmic solution to the GCS/SF problem*

This theoretical background can be summarized in the following macro-algorithm, in which the invariant clause for the loop is the existence of a set of non-redundant, consistent and multidimensional set of (constraint-generated) polynomials. In the event of the addition of new constraints to the scene, the algorithm converts them into polynomial(s), and tests their redundancy (by using Proposition 4), consistency (Proposition 2) and zero-dimensionality (Proposition 3). If the new constraint is redundant no action is taken; in the other two cases the invariant becomes false and the loop breaks. If the ideal has become zero-dimensional, a triangular Grobner Basis under some stated lexicographic order is extracted (Property 5) and solved (Property 3). Proposition 1 is the underlying basis of the algorithm, because it establishes that the $GB(F)$ faithfully represents F, with the same roots and Ideal set. In the algorithm presented below, the propositions or properties relevant to some important instructions are displayed at the left-hand side:

{Pre: W a fixed scenario }
$F = \{\}$
$GB_t = \{\}$
do new relation R_i
{Inv: F is consistent, non-redundant,

```
                     multidimensional}
                     R = R ∪ {R_i}
                     f = poly_form(W, R_i)
Proposition 2        if (1 ∈ GB_t(F ∪ {f})) then
                       stop (system is inconsistent)
                     else
Proposition 4          if (f ∈ Radical(F)) then
                         skip (f is redundant)
                       else
                         F = F ∪ {f}
Proposition 1            GB_t = GrobnerBasis(F, ≺_t)
Proposition 3            if (ZeroDimension(GB_t)) then
                           break loop
                         else
                           skip (next relation-constraint)
                         fi
                       fi
                     fi
                   od
Property 5         GB_l = GrobnerBasis(F, ≺_l)
Property 3         S = triangular_solution(GB_l)
                   {Post: R = {R_i} a set of relations;
                    S = feasible(W, R)}
```

The limitations of Grobner Basis, and for that matter any symbolic algebraic geometry method, in solving this problem is the explosive computational complexity of the method, and its still unexplored behavior in dealing with real arithmetic. If F is a set in $K[x_1, x_2, \ldots, x_n]$, with maximum exponent m, the Grobner Basis can contain polynomials of degree proportional to 2^{2^m} (Hoffmann, 1989).

3.3. *Double-peg-hole example: Grobner Basis for the constraint set*

This section continues the example introduced previously (see Fig. 2), to illustrate the execution of the proposed algorithm.

The basic condition of (dexterous) orthonormality of the *Rot* matrix produces the following set of equations:

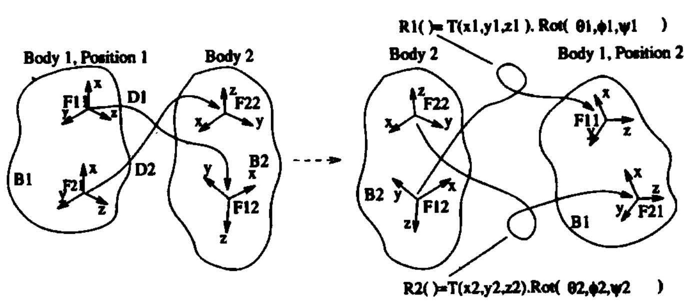

Fig. 3. Two-body example of canonical variable modeling of the GCS/SF problem.

$$\left.\begin{aligned}
{x_{11}}^2 + {x_{21}}^2 + {x_{31}}^2 - 1 &= 0;\\
{x_{12}}^2 + {x_{22}}^2 + {x_{32}}^2 - 1 &= 0;\\
{x_{13}}^2 + {x_{23}}^2 + {x_{33}}^2 - 1 &= 0;\\
x_{13}x_{12} + x_{23}x_{22} + x_{33}x_{32} &= 0;\\
x_{11}x_{12} + x_{21}x_{22} + x_{31}x_{32} &= 0;\\
x_{21}x_{32} - x_{31}x_{22} - x_{13} &= 0;\\
x_{31}x_{12} - x_{11}x_{32} - x_{23} &= 0;\\
x_{11}x_{22} - x_{21}x_{12} - x_{33} &= 0;
\end{aligned}\right\} \quad (5)$$

The lexicographic order used in this example for the calculation of the Grobner Basis is:

$$x_{11} \succ x_{12} \succ x_{13} \succ x_{21} \succ x_{22} \succ x_{23} \succ x_{31} \succ x_{32} \succ x_{33} \succ T_x \succ T_y \succ T_z \quad (6)$$

Constraint 1. $LN_1 - ON - LN_3$**: when this constraint is applied, the conditions**

$$(P_1' - P_3) \times v_3 = 0 \quad (P_1' \in LN_3); v_1' \times v_3 = 0 \quad (v_1' \parallel v_3) (7)$$

produce an initial polynomial set:

$$\{d + T_z = 0, n - T_y = 0, x_{31} = 0, -x_{21} = 0\}. \quad (8)$$

The Grobner Basis corresponding to this condition is shown below. The parameters of the World configuration (c, d, w, n) appear as constants in the basis.The $\underline{f}$ notation is used to stress the fact that f is *head*() of a polynomial:

$$\left.\begin{array}{lll}
\underline{T_z} + d = 0; & \underline{T_y} - n = 0; & \underline{x_{32}}^2 + {x_{33}}^2 - 1 = 0;\\
\underline{x_{31}} = 0; & \underline{x_{23}}^2 + {x_{33}}^2 - 1 = 0; & \underline{-x_{22}} + x_{22}{x_{33}}^2 - x_{23}x_{32}x_{33} = 0;\\
\underline{x_{22}x_{32}} + x_{23}x_{33} = 0; & \underline{x_{22}x_{23}} + x_{33}x_{32} = 0; & \underline{x_{22}}^2 - {x_{33}}^2 = 0;\\
\underline{x_{21}} = 0; & \underline{x_{13}} = 0; & \underline{x_{12}} = 0;\\
 & \underline{x_{11}} + x_{23}x_{32} - x_{22}x_{33} = 0. &
\end{array}\right\} (9)$$

First, the fact that the GB does not contain a term of the type $k = 0$, with k being any constant, suggests that it cannot be concluded that the constraint is inconsistent with the pre-existing scene. However, it would be possible to have a solution with complex numbers that is not physically realizable. Secondly, proposition 3 and property 4 rule out a zero-dimensional ideal because T_x and x_{33} are not *head*() of any polynomial as pure powers. The original constraint specification would continue being satisfied under subsequent rotations around the line LN_3 (x_{33}) and translations (T_x) along it. Consequently, there exists an infinite number of solution configurations. Although the translational degree of freedom is easily related to T_x, the relationship of the rotation to x_{33} is less intuitive.

Constraint 2. $LN_2 - ON - LN_4$: this second constraint results in the equations:

$$(P'_2 - P_4) \times v_4 = 0 \quad (P'_2 \in LN_4); \quad v'_2 \times v_4 = 0 \quad (v'_2 \parallel v_4). \quad (10)$$

The Grobner Basis for the accumulated constraints, once again, shows neither inconsistency nor zero-dimensionality:

$$\left.\begin{array}{lll} \underline{T_z} + d = 0; & \underline{T_y} - n = 0; & \underline{x_{33}} + 1 = 0; \\ \underline{x_{32}} = 0; & \underline{x_{31}} = 0; & \underline{x_{23}} = 0; \\ \underline{x_{22}}^2 - 1 = 0; & \underline{x_{21}} = 0; & \underline{x_{13}} = 0; \\ \underline{x_{12}} = 0; & \underline{x_{11}} + x_{22} = 0. & \end{array}\right\} \quad (11)$$

In this case T_x is effectively the only degree of freedom left. Two assembly modes are possible, by setting $x_{22} = \pm 1$.

Constraint 3. $P_1 - ON - P_3$: if this additional constraint is enforced it would be expected that all the degrees of freedom become fixed, leaving only a finite number of configurations (D transformations) satisfying the constraints. Indeed, the Grobner Basis has each variable in the head term of an equation of the basis and would therefore be zero-dimensional:

$$\left.\begin{array}{lll} \underline{T_z} + d = 0; & \underline{T_y} - n = 0; & \underline{T_x}^2 - c^2 + w^2 - 2w.T_x = 0; \\ \underline{x_{33}} + 1 = 0; & \underline{x_{32}} = 0; & \underline{x_{31}} = 0; \\ \underline{x_{23}} = 0; \ \underline{c.x_{22}} + w - T_x = 0; & & \underline{x_{21}} = 0; \\ \underline{x_{13}} = 0; & \underline{x_{12}} = 0; & \underline{x_{11}}\, c - w + T_x = 0. \end{array}\right\} \quad (12)$$

Constraint 4. $P_2 - ON - P_4$: this further constraining renders the system inconsistent, unless the new constraint is redundant. When $P_2 - ON - P_4$ is requested, the Grobner Basis shows a topological inconsistency. Topological inconsistencies in general vanish the solution space, except for a special set of values, i.e. for a very special point on the line L_4 (which is not P_4) to receive the point P_2. Under the constraint $P_2 - ON - P_4$ the Grobner Basis is indeed $GB = \{1\}$.

The four instances of $GB(F)$, sequentially calculated as constraints are added, demonstrate that the GCS/SF problem can be solved using the order $T_z, T_x, \ldots, x_{11}$. Eqns 12 are in triangular form under this ordering; x_{11}, the highest in the order, appears in only one equation, whereas T_x, which is lower, appears in a number of equations.

The large number of variables (twelve) and equations (twenty), compounds the high computational complexity of the Grobner Basis construction. In addition, the non-canonical variables are difficult to relate to the degrees of freedom of the solution to the GCS/SF problem. The following sections discuss a more 'natural' set of variables that addresses these problems.

4. The $SE(3)$ group and the GCS/SF problem

The set of Euclidean displacements in E^3, $SE(3)$, is a (noncommutative) group (Ledermann, 1953, 1973): if G_1, G_2 and G_3 represent displacements in $SE(3)$ and $\circ$ represents the composition operation, the following properties hold:

- Closure: two displacements G_1, G_2 applied in sequence produce a new displacement $G_3 = G_1 \circ G_2$.
- Existence of identity: I is the null displacement in $SE(3)$: $G \circ I = I \circ G = G$.
- Existence of inverse: $\forall G \in SE(3) \exists G^{-1} \in SE(3)$ s.t. $G \circ G^{-1} = G^{-1} \circ G = I$. The inverse, G^{-1}, restores the affected entity to the original position.
- Associativity: the effect of displacements is cumulative. The following sequences are identical: $(G_1 \circ G_2) \circ G_3 = G_1 \circ (G_2 \circ G_3)$.

$SE(3)$ presents subsets that are groups themselves, and that express certain common classes of displacements. They are called *subgroups*. For example, the subgroup of the rotations about a *given* axis u in the space Ru is a subset of $SE(3)$, and a group itself.

Given A, B, *subgroups* of the Euclidean group $SE(3)$, A is *conjugate* of B ($A \sim B$) iff $\exists\, T \in SE(3)$ such that $A = T^{-1}BT$. The relation $A \sim B$ is an equivalence relation. It is symmetric, reflexive and transitive, and defines equivalence classes called *conjugation classes*.

The T element above represents a rigid displacement. Therefore, two displacements A and B are equivalent iff a change of basis T converts one into the other. An equivalence class (in this case a *conjugation class*) represents the closure of a set of elements under the relation $A \sim B$. As a consequence, the class can be represented by an element with desirable properties called *canonical*. In our case, a very important property for a canonical form is that it be minimal in the number of variables that specify it. This implies that independence is a necessary condition for canonical sets of variables. Conjugation classes have a canonical subgroup that represents any other subgroup in the class by applying the transformation for a change of basis. A list of the conjugation classes for the subgroups of $SE(3)$ and their canonical representation (Herve, 1978), as well as their degrees of freedom is shown in Table 2. In this Table, $twix(\theta)$ means a rotation about the X axis by θ, $XTOY$ means a rotation by 90^o about the Z axis and $trans(x, y, z)$ indicates a general spatial translation. The concept of equivalence (conjugation) allows one to name certain displacements in $SE(3)$ as 'linear translations', 'rotations', 'planar slidings', etc, therefore relating the subgroups of $SE(3)$ with kinematic *constraints*. For example, 'rotations' are all transformations of the form

$$R_u(\theta) = B.\hat{R}_u.B^{-1} = B.twix(\theta).B^{-1}$$

Table 2. Conjugation classes and their canonical forms

Dof	*Symbol*	*Conjugation class*	*Canonical subgroup*
1	R_u	Rotations about axis u	$\{twix(\theta)\}$
1	T_u	Translations along axis u	$\{trans(x,0,0)\}$
1	$H_{u,p}$	Screw movement along axis u, with pitch p	$\{trans(x,0,0).twix(px)\}$
2	C_u	Cylindrical movement along axis u	$\{trans(x,0,0).twix(\theta)\}$
2	T_p	Planar translation parallel to plane P	$\{trans(0,y,z)\}$
3	G_p	Planar sliding along plane P	$\{trans(0,y,z).twix(\theta)\}$
3	S_o	Spherical rotation about center 'o'	$\{twix(\psi).XTOY.twix(\phi).XTOY.twix(\theta)\}$
3	T	3D translation	$\{trans(x,y,z)\}$
3	$Y_{v,p}$	Translating screw axis v, pitch p	$\{trans(x,y,z).twix(px)\}$
4	X_v	3D translation followed by rotation about v	$\{trans(x,y,z).twix(\theta)\}$

Table 3. Composition and intersection of trivial constraints

Groups	*Conditions on geometry*	*Intersection*	*Composition*
C_{u0}, C_{u1}	$u_0 \parallel u_1$	T_{u0}	$C_{u0} \circ R_{u1}$
T_{p0}, T_{p1}		T_{v0}; $v_0 = P_0 \cap P_1$	T

with $B \in SE(3)$. The displacement B represents the *geometric* part of a particular constraint, i.e. its dimensional information. The canonical part, $\hat{R}_u = twix(\theta)$, contains the *topological* information, i.e. the number and type of degrees of freedom. Given two subgroups G_1 and G_2 of $SE(3)$, the composition of the elements of G_1 and G_2, $G_1 \circ G_2$, produces what is called the *direct product* of groups G_1 and G_2. The intersection of the groups, which forms a subgroup of $SE(3)$ also, reflects the simultaneous application of constraints to the same geometric entity (Angeles, 1982; Herve, 1978; Thomas, 1991). In a kinematic situation the composition of constraints can be thought of as a *serial* arrangement of joints, while their intersection can be thought as a *parallel* one.

Sequences of constraints are expressed as: $R_1 \circ R_2 \circ R_3 \ldots \circ R_n$, with *trivial* constraints being defined as sequences of length 1. Herve (1978), provides tables with the results of composition and intersection of trivial constraints. Two such examples are shown in Table 3. Notice that the composition of trivial constraints in general produces one that is non-trivial ($C_{u0} \circ C_{u1}$, for example). This lack of closure of the subgroups of $SE(3)$ places a fundamental limitation in the reduction of constraint sets by using sequences of group compositions and intersections and rewrite procedures based on tables (such as Table 3). The goal of these procedures, not always attainable, is to reduce the whole constraint network to a single, trivial constraint relating two entities. Another limitation of this approach is its inability to deal with geometric inconsistencies. For example, the intersection of two cylindrical groups C_{u0} and C_{u1}, with $u_0 \parallel u_1$, produces a translational T_{u0} degree of freedom (Table 3). The physical situation corresponds to a rigidly linked pair of parallel pegs, entering into a pair of holes whose axes are also parallel. The method would correctly establish that a translational joint is left (topological result). However, the *distance* between the axes has still to be checked (geometric condition).

The canonical form of conjugation classes developed by Herve (1978) and given in Table 2 will be used to formulate the GCS/SF problem. The constraint reduction procedures will be supplemented by the more powerful algebraic geometric technique of the Grobner Basis. The following sections expand on this approach.

4.1. *Modeling methodology*

By definition, a *constraint* maintains invariant certain relations between the constrained entities. For example,a planar sliding, *Gp*, allows 2 translational and1 rotational degree of freedom, while still ensuring planar contact between the two parts (see Table 2). A rotational constraint, R_u, preserves axial and radial relative distances, allowing 1 angular degree of freedom between the constrained entities. Consequently, the contact constraints considered can be specified as shown in Table 4. For example, a $P-ON-PLN$ relation confines a point to be on a plane, therefore configuring a 5-dof constraint. It includes 2 dof related to the position of the point on the plane (T_P), and 3 dof, corresponding to the orientation (S) of the frame attached to the point. These (matrix) equations allow for the construction of the polynomial form of the GCS/SF problem. The methodology for this modeling is discussed next.

The *GCS/SF* problem is stated as a series of constraints R_i relating F_{i1} with F_{i2} as shown in Fig. 3 (corresponding to a two-body system). The $R_i()$ constraints are in general composed of translations $T()$ and rotations $Rot()$, as dictated by Tables 2 and 4. Body B_1 contains two features, whose frames are F_{11} and F_{21}. The corresponding features in body B_2 are F_{12} and F_{22}. The goal is to find a final position of B_1 (assuming B_2 stationary), such that F_{11} relates to F_{12} and F_{21} relates to F_{22} satisfying the invariances dictated by $R_1()$ and $R_2()$ respectively.

The final position of B_1 must be such that feature frames F_{11} and F_{12} differ exactly in the orientation and position changes allowed by constraint $R_1()$. The same should be true for F_{21} and F_{22} with regard to $R_2()$.The equations expressing the facts above are:

$$B_1.F_{11}.R_1() = B_2.F_{12}; \quad B_1.F_{21}.R_2() = B_2.F_{22}. \quad (13)$$

The above procedure can be generalized to the case in which there are several constraints $R_i()$ specified among bodies. Once the constraint equations are obtained by this procedure, the construction of the Grobner Basis and its interpretation are carried out in the manner described by the constraint management algorithm already discussed. The goal of the following sections is to apply the methodology just explained, and to evaluate its relative advantages with respect to non-canonical modeling.

5. Examples

A number of examples are presented next, using both canonical and non-canonical formulations. The first example continues the double-peg–hole configuration already modeled with non-canonical variables. Two more examples (3-body and 4-body) of increasing complexity are used to illustrate the behavior of the modeling and solution strategies with respect to the size of the problem.

5.1. *Double-peg–hole example: solution with canonical variables*

The group-theoretical methodology discussed in the previous section will be applied to the configuration of Fig. 2, in which the simultaneous enforcement of two $LN-ON-LN$ constraints is shown. The methodology mentioned above and the model provided by (13) result in:

$$F_{11}.C_{u1}(\theta_1, x_1).F_{12}^{-1} = F_{21}.C_{u2}(\theta_2, x_2).F_{22}^{-1} \quad (14)$$

This matrix equation can be expanded in the form

$$F_{11}.\begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & c_1 & -s_1 & 0 \\ 0 & s_1 & c_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.F_{12}^{-1} = F_{21}.\begin{bmatrix} 1 & 0 & 0 & x_2 \\ 0 & c_2 & -s_2 & 0 \\ 0 & s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.F_{22}^{-1} \quad (15)$$

where $c_1 = \cos(\theta_1); s_1 = \sin(\theta_1)$. Also, two equations of the form $c_1^2 + s_1^2 - 1 = 0$ are included. Eqn (14) represents a set of 12 polynomials and 6 variables (c is a constant). This equation set has the following (lexicographic) Grobner Basis:

$$\underline{s_2} - 1 = 0; \quad \underline{c_2} = 0; \quad \underline{s_1} + 1 = 0; \quad \underline{c_1} = 0;$$
$$\underline{x_1} - x_2 + c = 0; \quad (16)$$

which is based on the order: $x_1 \succ x_2 \succ c_1 \succ s_1 \succ c_2 \succ s_2$. The absence of a polynomial whose *head*() term contains a pure power of x_2 indicates that the ideal is not zero-dimensional and that x_2 is a free variable. It is also known from its role in the canonical formulation that it represents a translational degree of freedom. Being fewer than non-canonical ones, canonical variables present advantages in computing effort during the construction of the Grobner Basis. Table 7 presents some statistics corresponding to the examples developed.

5.2. *Three-body example*

This example compares the performance of canonical and non-canonical formulations with a larger set of constraints and bodies. Fig. 4 shows the scenario being modeled (adapted from Thomas and Torras (1989)),with the imposed constraints appearing in Table 5. Columns 2 and 3 in Table 5 show the parametric expression of the features F_{ij}. Column 4 presents the canonical groups involved in the constraint, with the corresponding degrees of freedom. Notice that, although the constraints imposed between the entities are trivial, the fact that length-3 chains appear in the formulation makes the

Table 4. Entity relations in the form of kinematic joints

Contact constraint	*Joint chain*	*Kinematic joints in chain*	*Dof*
$P-ON-P$	S_o	Spherical	3
$P-ON-LN$	$T_v \circ S$	Linear translation, spherical	4
$P-ON-PLN$	$T_P \circ S$	Planar translation, spherical	5
$LN-ON-LN$	C_v	Cylindrical	2
$LN-ON-PLN$	$T_P \circ R_v \circ R_w$	Planar translation, revolute	4
$PLN-ON-PLN$	$T_P \circ R_v$	Planar translation, revolute	3

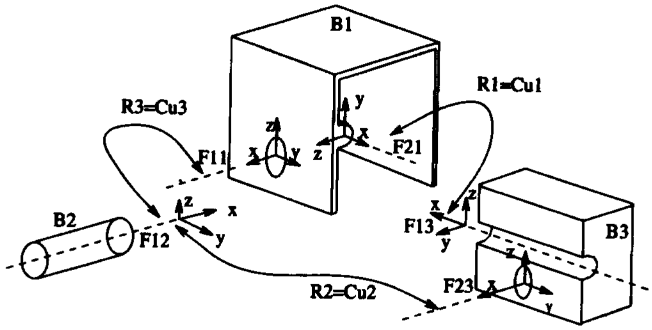

Fig. 4. Three-body assembly.

direct application of group intersection impossible, while the algebraic-geometry analysis proposed in this investigation remains applicable.

5.2.1. *Solution for three-body example using non-canonical formulation*

Assuming that the (coordinate frame of) body B_1 coincides with the World Coordinate System, bodies B_2, B_3 are in (unknown) positions $D2$ and $D3$ respectively. The constraints mentioned above result in the following equations ($D2$ and $D3$ are orthonormal):

$$\left.\begin{array}{ll} v_{21} \times v_{13} = 0; & v_{21} \times (p_{21} - p_{13}) = 0; \\ v_{12} \times v_{11} = 0; & v_{12} \times (p_{12} - p_{11}) = 0; \\ v_{12} \times v_{23} = 0; & v_{12} \times (p_{12} - p_{23}) = 0; \\ \det(D2) = +1; & \det(D3) = +1. \end{array}\right\} \quad (17)$$

This system results in the following Grobner Basis ($D2 = \{D2_{i,j}\}$ and $D3 = \{D3_{i,j}\}$), which is calculated based on the ordering $D2_{11} \succ D2_{12} \succ D2_{13} \succ D2_{14} \succ D2_{21} \succ D2_{22} \succ D2_{23} \succ D2_{24} \succ D2_{31} \succ D2_{32} \succ D2_{33} \succ D2_{34} \succ D3_{11} \succ D3_{12} \succ D3_{13} \succ D3_{14} \succ D3_{21} \succ D3_{22} \succ D3_{23} \succ D3_{24} \succ D3_{31} \succ D3_{32} \succ D3_{33} \succ D3_{34}$:

$$\left.\begin{array}{lll} \underline{D3_{34}} = 0; & \underline{D3_{33}} - 1 = 0; & \underline{D3_{32}} = 0; \\ \underline{D3_{31}} = 0; & \underline{D3_{24}^2} - 10\,D3_{24} = 0; & \underline{D3_{23}} = 0; \\ \underline{5\,D3_{22}} - 5 + D3_{24} = 0; & \underline{D3_{21}} = 0; & \underline{5\,D3_{14}} + D3_{24} = 0; \\ \underline{D3_{13}} = 0; & \underline{D3_{12}} = 0; & \underline{5\,D3_{11}} - 5 + D3_{24} = 0; \\ \underline{D2_{34}} - 2 = 0; & \underline{D2_{32}{}^2} + D2_{33}{}^2 - 1 = 0; & \underline{D2_{31}} = 0; \\ \underline{D2_{24}} - 5 = 0; & \underline{D2_{23}{}^2} + D2_{33}{}^2 - 1 = 0; & \\ \multicolumn{3}{l}{\underline{-D2_{22}} - D2_{23}\,D2_{32}\,D2_{33} + D2_{22}\,D2_{33}{}^2 = 0;} \\ \underline{D2_{22}\,D2_{32}} + D2_{23}\,D2_{33} = 0; & \underline{D2_{22}\,D2_{23}} + D2_{32}\,D2_{33} = 0; & \underline{D2_{22}{}^2} - D2_{33}{}^2 = 0; \\ \underline{D2_{21}} = 0; & \underline{D2_{13}} = 0; & \underline{D2_{12}} = 0; \\ \multicolumn{3}{l}{\underline{D2_{11}} + D2_{23}\,D2_{32} - D2_{22}\,D2_{33} = 0.} \end{array}\right\} \quad (18)$$

The Grobner Basis (shown in (18)) is presented in triangular form; it shows that variables $D2_{14}$ and $D2_{33}$ are missing in the *head*() terms of polynomials. Although this information is not self-evident from the solution set, these two variables have very definite roles in the $D2$ matrix: $D2_{14}$ represents a translational degree of freedom, whereas $D2_{33}$ represents a rotational degree of freedom about an unknown axis in the space, determined by the eigenvalues and eigenvectors of the submatrix $Rot_2 = D2_{ij} (i = 1 \ldots 3, j = 1 \ldots 3)$ (Bottema and Roth, 1979). The solution implies (as expected) that body B_3 is fixed while body B_2 still has degrees of freedom left, represented by the variables $D2_{14}, D2_{33}$. Notice that, in this case, non-instantiation of $D2_{33}$ immediately spreads to $D2_{32}, D2_{23}, D2_{22}$, because these values control the eigenvalues and eigenvectors of the matrix Rot_2.

5.2.2. *Solution for three-body example using canonical formulation*

In this case the system of group-based matrix equations can be stated as

$$F_{21}.C_{u1}(x_1, \theta_1).F_{13}^{-1} = F_{11}.C_{u3}(x_3, \theta_3).C_{u2}(x_2, \theta_2).F_{23}^{-1} \quad (19)$$

Table 5. Constraint specification for three-body example

Constraint	*Feature*	*Feature*	*Canonical subgroup*
$R_1 : LN - ON - LN$	$F_{21} = (p_{21}, v_{21})$	$F_{13} = (p_{13}, v_{13})$	$C_u(x_1, \theta_1)$
$R_2 : LN - ON - LN$	$F_{23} = (p_{23}, v_{23})$	$F_{12} = (p_{12}, v_{12})$	$C_u(x_2, \theta_2)$
$R_3 : LN - ON - LN$	$F_{11} = (p_{11}, v_{11})$	$F_{12} = (p_{12}, v_{12})$	$C_u(x_3, \theta_3)$

Table 6. Constraint specification for four-body example

Constraint	*Feature*	*Feature*	*Canonical subgroup*
$R_1 : LN - ON - LN$	$F_{21} = (p_{21}, v_{21})$	$F_{13} = (p_{13}, v_{13})$	$C_u(x_1, \theta_1)$
$R_2 : LN - ON - LN$	$F_{31} = (p_{31}, v_{31})$	$F_{12} = (p_{12}, v_{12})$	$C_u(x_2, \theta_2)$
$R_3 : LN - ON - LN$	$F_{22} = (p_{22}, v_{22})$	$F_{23} = (p_{23}, v_{23})$	$C_u(x_3, \theta_3)$
$R_4 : LN - ON - LN$	$F_{14} = (p_{14}, v_{14})$	$F_{32} = (p_{32}, v_{32})$	$C_u(x_4, \theta_4)$
$R_5 : LN - ON - LN$	$F_{14} = (p_{14}, v_{14})$	$F_{33} = (p_{33}, v_{33})$	$C_u(x_5, \theta_5)$
$R_6 : LN - ON - LN$	$F_{14} = (p_{14}, v_{14})$	$F_{11} = (p_{11}, v_{11})$	$C_u(x_6, \theta_6)$

Using an ordering $x_1 \succ x_2 \succ x_3 \succ s_1 \succ c_1 \succ s_2 \succ c_2 \succ s_3 \succ c_3$ produces a (lexicographic) Grobner Basis:

$$\left.\begin{array}{lll} \underline{s_3}^2 - 1 + c_3^2 = 0; & \underline{c_2} - c_3 = 0; & \underline{s_2} + s_3 = 0; \\ \underline{c_1} - 1 = 0; & \underline{s_1} = 0; & \underline{x_2} + x_3 = 0; \\ \underline{x_1} = 0. & & \end{array}\right\} \quad (20)$$

The analysis of this basis shows variables x_1, c_1, s_1 as completely instanced, indicating that the position of the body B_3 is fixed. Body B_2 is free to rotate about and translate along axis F_{12} because x_3 and c_3 appear as free variables. This direct relation between the rotational and translational degrees of freedom of B_2 with the variables x_3 and c_3 is enabled by the use of canonical formulation, which also decreases significantly the size of the problem (see Table 7).

5.3. *Four-body example*

This example, whose scenario is shown in Fig. 5 (adapted from Thomas and Torras (1989)), introduces an additional entity to the three-body example just discussed (a different body numbering is used). The set of constraints imposed is presented in Table 6.

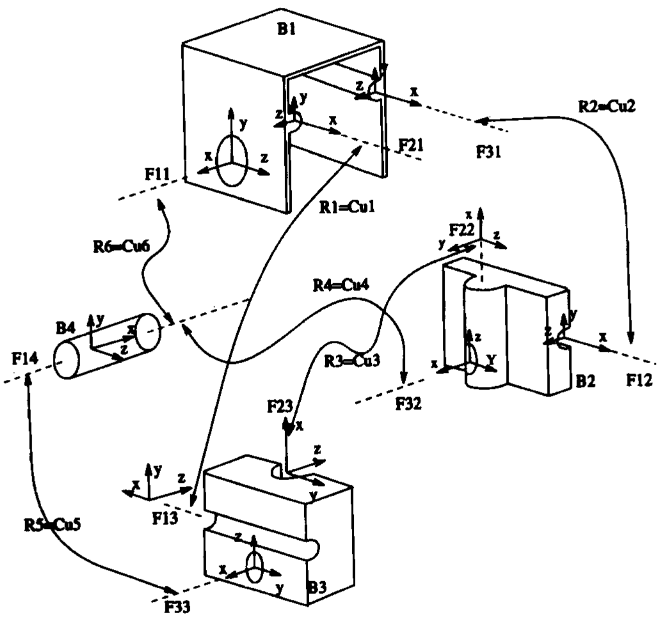

Fig. 5. Four-body assembly.

5.3.1. *Solution for four-body example using non-canonical formulation*

For this formulation, body B_1 is assumed to coincide with the World Coordinate System, with bodies B_2, B_3 and B_4 occupying positions expressed by the unknown, rigid transformations matrices $D2$, $D3$ and $D4$ respectively.

The statement of the non-canonical formulation of the problem is carried out as discussed before. In order to calculate the Grobner Basis for the corresponding polynomial ideal, the ordering of the variables used is $D4_{11} \succ D4_{12} \succ D4_{13} \succ D4_{14} \succ D4_{21} \succ D4_{22} \succ D4_{23} \succ D4_{24} \succ D4_{31} \succ D4_{32} \succ D4_{33} \succ D4_{34} \succ D3_{11} \succ D3_{12} \succ D3_{13} \succ D3_{14} \succ D3_{21} \succ D3_{22} \succ D3_{23} \succ D3_{24} \succ D3_{31} \succ D3_{32} \succ D3_{33} \succ D3_{34} \succ D2_{11} \succ D2_{12} \succ D2_{13} \succ D2_{14} \succ D2_{21} \succ D2_{22} \succ D2_{23} \succ D2_{24} \succ D2_{31} \succ D2_{32} \succ D2_{33} \succ D2_{34}$.

The lexicographic Grobner Basis for this particular ordering is

$$\left.\begin{array}{lll} \underline{D2_{34}} = 0; & \underline{D2_{33}} - 1 = 0; & \underline{D2_{32}} = 0; \\ \underline{D2_{31}} = 0; & \underline{D2_{24}} = 0; & \underline{D2_{23}} = 0; \\ \underline{D2_{22}} - 1 = 0; & \underline{D2_{21}} = 0; & \underline{D2_{14}} + 12 = 0; \\ \underline{D2_{13}} = 0; & \underline{D2_{12}} = 0; & \underline{D2_{11}} - 1 = 0; \\ \underline{D3_{34}} = 0; & \underline{D3_{33}} - 1 = 0; & \underline{D3_{32}} = 0; \\ \underline{D3_{31}} = 0; & \underline{D3_{24}} = 0; & \underline{D3_{23}} = 0; \\ \underline{D3_{22}} - 1 = 0; & \underline{D3_{21}} = 0; & \underline{D3_{14}} = 0; \\ \underline{D3_{13}} = 0; & \underline{D3_{12}} = 0; & \underline{D3_{11}} - 1 = 0; \\ \underline{D4_{34}} - 2 = 0; & \underline{D4_{32}}^2 + D4_{33}^2 - 1 = 0; & \underline{D4_{31}} = 0; \\ \underline{D4_{24}} - 5 = 0; & \underline{D4_{23}}^2 + D4_{33}^2 - 1 = 0; & \\ \multicolumn{3}{l}{\underline{-D4_{22}} - D4_{23}\,D4_{32}\,D4_{33} + D4_{22}\,D4_{33}^2 = 0;} \\ \underline{D4_{23}\,D4_{33}} + D4_{22}\,D4_{32} = 0; & \underline{D4_{22}\,D4_{23}} + D4_{32}\,D4_{33} = 0; & \underline{D4_{22}}^2 - D4_{33}^2 = 0; \\ \underline{D4_{21}} = 0; & \underline{D4_{13}} = 0; & \underline{D4_{12}} = 0; \\ \multicolumn{3}{l}{\underline{D4_{11}} + D4_{23}\,D4_{32} - D4_{22}\,D4_{33} = 0.} \end{array}\right\} \quad (21)$$

The analysis of this basis shows that there are no polynomials with *head*() terms containing pure powers of $D4_{33}$ and $D4_{14}$, indicating that two degrees of freedom are left in the scene. As discussed in the three-body example, the rod B_4 is free to undergo linear ($D4_{14}$) as well as rotational ($D4_{33}$) movement about its symmetry axis.

Table 7. Statistics for examples: non-canonical versus canonical variables

Example	*Variable type*	*Variables*	*Equations*	*GB size*	*Time (seconds)*
Double-peg–hole example	Non-canonical	12	20	16	1.53
Double-peg–hole example	Canonical	6	14	6	0.25
Three-body example	Non-canonical	20	30	24	6.08
Three-body example	Canonical	9	15	7	0.51
Four-body example	Non-canonical	36	54	37	39.01
Four-body example	Canonical	18	42	16	2.16

5.3.2. *Solution for four-body example using canonical formulation*

The following canonical equations carry the topological and geometrical information of the example. Notice that, unlike the previous example, three loops of constraints have to be simultaneously satisfied.

$$\left.\begin{array}{l} F_{11}.R_6(x_6,\theta_6).F_{14}^{-1}.F_{14}.R_4(x_4,\theta_4).F_{32}^{-1} = F_{31}.R_2(x_2,\theta_2).F_{12}^{-1} \\ F_{11}.R_6(x_6,\theta_6).F_{14}^{-1}.F_{14}.R_5(x_5,\theta_5).F_{33}^{-1} = F_{21}.R_1(x_1,\theta_1).F_{13}^{-1} \\ F_{14}.R_4(x_4,\theta_4).F_{32}^{-1}.F_{22}.R_3(x_3,\theta_3).F_{23}^{-1} = F_{14}.R_5(x_5,\theta_5).F_{33}^{-1} \end{array}\right\} \quad (22)$$

In this case, the variable ordering used is: $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6 \succ c_1 \succ s_1 \succ c_2 \succ s_2 \succ c_3 \succ s_3 \succ c_6 \succ s_6 \succ c_4 \succ s_4 \succ s_5 \succ c_5$, which produces the following lexicographic Grobner Basis:

$$\left.\begin{array}{lll} \underline{s_5}^2 + c_5^2 - 1 = 0; & \underline{s_4} - c_5 = 0; & \underline{c_4} + s_5 = 0; \\ \underline{s_6} + s_5 = 0; & \underline{c_6} - c_5 = 0; & \underline{s_3} = 0; \\ \underline{c_3} - 1 = 0; & \underline{s_2} = 0; & \underline{c_2} - 1 = 0; \\ \underline{s_1} = 0; & \underline{c_1} - 1 = 0; & \underline{x_5} + x_6 = 0; \\ \underline{x_4} + 6 + x_6 = 0; & \underline{x_3} = 0; & \underline{x_2} + 10 = 0; \\ \underline{x_1} = 0. & & \end{array}\right\} \quad (23)$$

This triangular Grobner Basis presents polynomials with every variable as pure power in the *head*() term, except for x_6 and c_5. x_6 is the translational degree of freedom of body B_4, and variable c_5 represents its rotational degree of freedom. Therefore the example might have given s_5, s_6 or c_6 as free variable instead of c_5, because they are equivalent with respect to the degree of freedom that they represent. The computational expenses of this example are presented in Table 7.

6. Discussion of examples

Although in the examples shown, the size of the Grobner Basis correlates with the size of the original system, it is expected that in more complex scenes, a large Grobner Basis will be produced, with exponentially growing computing resources allocated to the problem (Hoffmann, 1989; Becker, 1993). Partially compensating this inherent difficulty of the GCS/SF problem, the use of canonical variables seems to result in smaller problem sizes and correspondingly smaller Grobner Bases.

In situations with a large number of bodies in the scene, it is not practicable to place the emphasis of modeling on the parametric expression of the entities (non-canonical). Instead, a degree-of-freedom (canonical) modeling would produce a smaller problem size. This is because the compactness of the canonical representation would become more notorious as the problem size grows. In contrast, in situations with a small number of bodies and many interactions between them, choosing a representation that grows with each constraint added (canonical), would present disadvantages. In such situations the size of a non-canonical set of variables is small and constant, regardless of the number of constraints added. Therefore such a formulation of the problem would be more convenient.

For purposes of automated reasoning, the advantage of canonical variables is evident, because they directly represent the degrees of freedom of the entities involved. This characteristic makes them especially attractive in kinematic analysis and design and in assembly planning. In the non-canonical statement the physical role of the variables is obscured (or shared) by other variables, therefore making the automated reasoning more difficult.

Further investigation is needed, to characterize which systems of constraints are efficiently modeled by each method.

7. Conclusions

Satisfaction of geometric constraints plays an essential part in design and assembly planning. For example, constraint-based design systems must, at a minimum, be able to handle geometric constraints. Further, if manufacturing plans are to be automatically updated with changes in design, the problem becomes very relevant in areas such as assembly, process and inspection planning.

In this investigation the problem of reasoning about geometric constraints was addressed using Grobner Bases. The Grobner Basis of a polynomial set $F = \{p_1, p_2, \ldots, p_n\}$ has several properties for characterization of the variety of its polynomial ideal. From the GCS/SF problem perspective, these properties have enabled the identification of remaining spatial degrees of freedom in a given scenario and the assessment of redundancy and (in)consistency within a constraint set F. An algorithmic procedure for exploiting the properties of Grobner Bases in a design/planning environment was given and illustrated by several examples. Grobner Bases provide a framework for the integral treatment of topological and geometrical consistency in the set of constraints. Such a framework is not limited to trivial constraints as with techniques associated with approaches applying rewriting rules based on exclusively topological considerations.

A drawback of the direct application of algebraic geometry techniques is the growth of computational effort with problem size. This was demonstrated in a series of examples of increasing complexity. This problem is addressed in this investigation by the application of a set of variables derived from the subgroups of the special group of Euclidean displacements $SE(3)$. These so-called *canonical* variables provide a compact and physically meaningful representation of the GCS/SF problem. Therefore they facilitate the automated interpretation and analysis of the solution in terms of the degrees of freedom of the entities involved.

Future work addresses the preprocessing of the topological part of the constraint network by applying group-reduction techniques of Angeles (1988), Herve (1978) and Thomas (1991), and by using the Grobner basis method with a reduced set of constraints.

In this investigation the primary focus has been on the feasibility of a set of contact constraints defined on a set of entities. For a complete analysis of the GCS/SF problem, the question of interference has to be addressed, in which the configuration has to be tested for invasion of volumes among bodies. This problem is not a trivial extension of what is reported here, because interference constraints reduce to sets of inequalities. Beyond the problem of interference, the problem of reachability of a kinematically valid, non-invasive proposed world has to be studied. In that problem the path of the objects to reach the desired configuration without collisions becomes a relevant subject.

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