Basic Notions of Point Set Topology, Metric Spaces and the PL-category

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Abstract

This article presents definitions and examples of concepts regarding metric and topological spaces and simplicial complexes. Such mathematical concepts constitute an important part of the formal grounds of software applications in engineering, involving curves,, surfaces or solids in dimensions two or three respectively.

Key words: metric spaces, topological spaces, simplicial complexes, piecewise-linear category.

1 Introduction

The concepts described in this article constitute the mathematical bases of most CAD/CAM/CAE software packages available nowadays. Such concepts include metric spaces, point-set topology and simplicial complexes (also referred as the PL-category). Several sources were consulted [5,1,2,4,3,6] for collecting the definitions and examples presented in this article.

The intended audience of this article is in first place under-grade students of computer science and mechanical engineering involved in the usage and development of CAD/CAM/CAE applications, but it also attain any person who is interested in learning the conceptual basis of such applications, for either improving his comprehension, developing related software applications and/or becoming familiar with the language used in the scientific literature.

2 Metric spaces

Definition 1 (Metric) *Let X be an non-empty set. A* metric *on X is a function $d\colon X\times X\to\mathbb{R}$ which satisfies the following conditions for each pair $x,y\in X$:*

(1) $d(x,y)\geq 0$
(2) $d(x,y)=0 \Leftrightarrow x=y$
(3) $d(x,y)=d(y,x)$ *(symmetry).*
(4) $d(x,y)\leq d(x,z)+d(z,y)$ *(triangle inequality).*

The number $d(x,y)$ is called the distance *between x and y.*

Definition 2 (Metric space) *A* metric space *consists of a pair (X,d) where X is a non-empty set and d is a metric for X. Whenever it makes no confusion, a metric space (X,d) is denoted by its underlying set X.*

Usually, an element $x\in X$ is referred as a *point* of the metric space (X,d). The examples presented below show that a metric can be defined for any non-empty set, regardless whether its elements are numbers or any other kind of objects.

Example 1 Let X be an arbitrary non-empty set, and d a function defined by

$$d(x,y)=\begin{cases}0 \text{ if } x=y\\ 1 \text{ if } x\neq y\end{cases}$$

This definition yields to the metric space (X,d) since d satisfies the conditions of a metric. The metric d is called the *discrete metric* on X.

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$d(x,y)$	a	b	c	d	e	f
a	0	3	2	3	4	4
b	3	0	1	1	3	1
c	2	1	0	1	4	2
d	3	1	1	0	3	1
e	4	3	4	3	0	2
f	4	1	2	1	2	0

Table 1
Metric defined by $d : X \to \mathbb{Z}^+$, with $X = \{a, b, c, d, e, f\}$

Example 2 Let $n \geq 1$ be an integer, and let

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}.$$

The function

$$d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2}$$

defines a metric for $\mathbb{R}^n$. $\mathbb{R}^0$ is defined as a set $\{\mathcal{O}\}$ with a single element.

Example 3 Let

$$X = \{f : [0,1] \to \mathbb{R} : f \text{ is a continuous function}\}.$$

The function

$$d(f(x), g(x)) = \max(f(x), g(x))$$

for $x \in [0,1]$ defines a metric on X.

Example 4 Let $X = \{a, b, c, d, e, f\}$, and d a function defined in table 1. It can be verified that the function $d : X \to \mathbb{Z}^+$ defines a metric on X since it satisfies the three conditions of a metric.

Definition 3 (Open ball) *Let (X, d) a metric space. Let $x_0 \in X$ and $r > 0$. The* open ball *with* center x_0 *and* radius r *is the subset of X defined by*

$$B_r(x_0) = \{x : d(x, x_0) < r\}.$$

In example 4, the open ball $B_4(a)$ is the set $\{a, b, c, d\}$ which include all the points at distance strictly smaller than 4 between the points of X and a. The open ball $B_1(b)$ is the single point b.

Definition 4 (Open set) *Given a metric space X, a set $G \subset X$ is said to be* open *if for each $x \in G$ there exists a $r_x > 0$ such that $B_{r_x}(x) \subset G$.*

Given a metric space X, the following predicates regarding open sets are satisfied:

(1) The empty set $\emptyset$ and the full space X are open sets.
(2) Each open ball in X is an open set.
(3) The union of an arbitrary collection of open sets in X is open.
(4) The intersection of a finite collection of open sets in X is open.

Definition 5 (Interior) *Let be X a metric space, and let $A \subset X$. A point $x \in A$ is called an* interior point *of A if*

$$(\exists r > 0)(B_r(x) \subset A).$$

The interior *of A, denoted by* $\text{Int}(A)$, *is the set defined by all the interior points of A.*

The basic properties of the interior operation are the following:

(1) $\text{Int}(A) \subset A$.
(2) $\text{Int}(A)$ is an open set.
(3) A is an open set $\Leftrightarrow A = \text{Int}(A)$.
(4) $\text{Int}(A) = \bigcup_i G_i$, where $G_i \subset A$ and G_i is open, i.e. $\text{Int}(A)$ is the largest open subset of A.

Example 5 The interior of the half-open interval $[0,1) \subset \mathbb{R}$ is the open interval $(0,1)$.

Definition 6 (Limit point) *Let X be a metric space and $A \subset X$. A point $x \in X$ is called a* limit point *of A if*

$$(\forall r > 0)(\exists w \in B_r(x))(w \in A \land w \neq x))$$

Example 6 The set $\{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ has 0 as a limit point, and it is not in the set. Furthermore, 0 is its only limit point.

Definition 7 (Closed set) *Let X be a metric space. A set $F \subset X$ is said to be a* closed set *if it contains each one of its limits points.*

Example 7 The interval $[-1, 0) \subset \mathbb{R}$ is not a closed set since it does not contain the limit point 0.

Example 8 Let X be a non-empty set, and $x \in X$. Under the discrete metric, the closed ball $B_r[x] = \{x\}$ when $r < 1$, and $B_r[x] = X$ when $r \geq 1$.

Definition 8 (Closed ball) *Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. The* closed ball $B_r[x_0]$ *with center x_0 and radius r is defined by*

$$B_r[x_0] = \{x : d(x, x_0) \leq r\}.$$

Given a metric space X, the following predicates regarding closed sets are satisfied:

(1) The empty set $\emptyset$ and the full space X are closed sets.

(2) $F \subset X$ is closed $\Leftrightarrow F'$ is open.
(3) The intersection of an arbitrary collection of closed sets in X is closed.
(4) The union of a finite collection of closed sets in X is closed.

Definition 9 (Closure) *Let X be a metric space and $A \subset X$. The* closure *of A, denoted by* $\text{Cl}(A)$ *or* $\bar{A}$, *is defined as the union of A and the set of all its limit points.*

The basic properties of the closure operation are the following:

(1) $A \subset \text{Cl}(A)$.
(2) $\text{Cl}(A)$ is a closed set.
(3) A is a closed set $\Leftrightarrow A = \text{Cl}(A)$.
(4) $\text{Cl}(A) = \bigcap_i F_i$, where $A \subset F_i$ and F_i is closed, i.e. $\text{Cl}(A)$ is the smallest closed superset of A.

Example 9 The closure of the half-open interval $[0, 1) \subset \mathbb{R}$ is $[0, 1]$ and the closure of the set $[0, 1) \cup (1, 2) \cup (2, 3] \subset \mathbb{R}$ is the closed interval $[0, 3]$.

Example 10 The closure of the set of rational numbers is the reals, i.e. $\bar{\mathbb{Q}} = \mathbb{R}$.

Definition 10 (Boundary) *Let X be a metric space and $A \subset X$. A point $x \in A$ is called a* boundary point *of A if*

$$(\forall r > 0)(B_r(x) \cap A \neq \emptyset \wedge B_r(x) \cap A' \neq \emptyset)$$

The boundary *of A, denoted by* $\text{Bd}(A)$, *is the set of all of its boundary points.*

The boundary operation has the following properties:

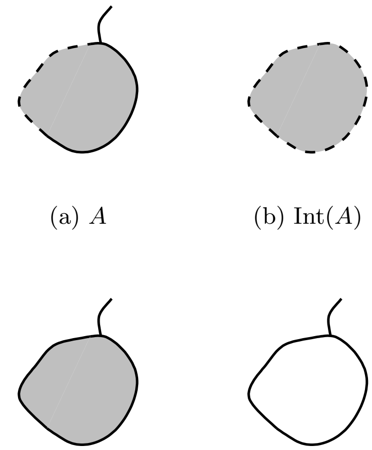
(1) $\text{Bd}(A) = \text{Cl}(A) \cap \text{Cl}(A')$.
(2) $\text{Bd}(A)$ is a closed set.
(3) A is closed $\Leftrightarrow \text{Bd}(A) \subset A$.
(4) $\text{Int}(A) \cap \text{Bd}(A) = \emptyset$.
(5) $X = \text{Int}(A) \cup \text{Bd}(A) \cup \text{Int}(A')$, and these sets are pairwise disjoint.

In figure 1, a subset A of $\mathbb{R}^2$ is depicted together with its corresponding interior, closure and boundary. There, heavy lines and shadowed regions denote points belonging to A and dashed lines and white regions denote sets of points not belonging to A.

Definition 11 (Convergence) *Let (X, d) be a metric space, and let*

$$\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$$

be a sequence of points in X. The sequence $\{x_n\}$ is convergent *if*



(a) A (b) $\text{Int}(A)$

(c) $\text{Cl}(A)$ (d) $\text{Bd}(A)$

Fig. 1. Example of the interior, closure and boundary of a set $A \subset \mathbb{R}^2$

(1) $(\exists x \in X)(\forall \epsilon > 0)(\exists n_0 \in \mathbb{Z}^+)(n \geq n_0 \Rightarrow d(x_n, x) < \epsilon)$ *or equivalently,*
(2) $(\exists x \in X)(\forall \epsilon > 0)(\exists n_0 \in \mathbb{Z}^+)(n \geq n_0 \Rightarrow x_n \in B_\epsilon(x))$.

The point x is called the limit *of the sequence $\{x_n\}$ and it is denoted by* $\lim x_n = x$.

If a sequence has a limit point it is unique. This justifies the last sentence in the previous definition.

Definition 12 (Continuous mapping) *Let (X, d_x) and (Y, d_y) be metric spaces and $f : X \to Y$. f is said to be* continuous at a point $x_0 \in X$ *if either*

(1) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)(d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon)$, *or equivalently*
(2) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)(f(B_\delta(x_0)) \subset B_\epsilon(f(x_0)))$.

The mapping $f : X \to Y$ is said to be continuous *if it is continuous at every point of X.*

3 Euclidean space

Definition 13 (Addition, scalar multiplication) *The* addition *and* scalar multiplication *in $\mathbb{R}^n$ are defined by*

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \alpha x &= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n). \end{aligned}$$

for each $x, y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.

It is easy to see that the addition and the scalar multiplication satisfy the following properties:

(1) $x + y = y + x$
(2) $x + (y + z) = (x + y) + z$
(3) There is an element $\mathcal{O}$ in $\mathbb{R}^n$ such that $x + \mathcal{O} = x$ for each $x \in \mathbb{R}^n$.
(4) For each $x \in \mathbb{R}^n$ there exists an element $-x$ such that $x + (-x) = \mathcal{O}$.
(5) $\alpha(x + y) = \alpha x + \alpha y$
(6) $(\alpha + \beta)x = \alpha x + \beta x$
(7) $(\alpha\beta)x = \alpha(\beta x)$
(8) $1 \cdot x = x$

Definition 14 (Euclidean norm) *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The* Euclidean norm *on $\mathbb{R}^n$, denoted by $\|x\|$, is defined by*

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

Definition 15 (Euclidean distance) *Let $x, y \in \mathbb{R}^n$. The* Euclidean distance *between x and y is defined as $\|x - y\|$.*

Definition 16 (n-dimensional Euclidean space) *Let be n a positive integer. $\mathbb{R}^n$ normed with the Euclidean norm is called the* n-dimensional Euclidean space*.*

Definition 17 (Subspace of $\mathbb{R}^n$) *. Let $S \subset \mathbb{R}^n$ be non-empty. S is said to be a subspace of $\mathbb{R}^n$ if it is closed under addition and scalar multiplication. More precisely, the following two conditions hold:*

(1) $u, v \in S \Rightarrow u + v \in S$
(2) $\lambda \in \mathbb{R}, u \in S \Rightarrow \lambda u \in S$

Conditions 1 and 2 guarantee that addition and scalar multiplication restrict as internal operations of S. It can be verified that S together with addition and multiplication by scalar satisfies the same eight properties addition and scalar multiplication satisfy in $\mathbb{R}^n$.

Definition 18 (Affine subspace of $\mathbb{R}^n$) *Let $S \subset \mathbb{R}^n$. S is said to be an affine subspace of $\mathbb{R}^n$ if for some fixed element $s_0 \in S$ (and therefore for any) the set $\{s - s_0 : s \in S\}$ is a subspace of $\mathbb{R}^n$.*

Definition 19 (Dimension of a affine subspace of $\mathbb{R}^n$) *Let $S \subset \mathbb{R}^n$ be an affine subspace. The dimension of S, denoted by $\dim(S)$, is defined as the dimension of the subspace $\{s - s_0 : s \in S\}$ for any $s_0 \in S$.*

Remember that the dimension of a subspace of $\mathbb{R}^n$ is the number of elements in any of its bases.

4 Point-set topology

Definition 20 (Topology) *Let X be a non-empty set. A collection $\mathcal{T}$ of subsets of X is called a* topology *if it satisfies the following three conditions:*

(1) $\emptyset \in \mathcal{T}$ *and* $X \in \mathcal{T}$.
(2) The union of an arbitrary collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$, or equivalently, if $\{U_i : i \in I\}$ is a collection such that $U_i \in \mathcal{T}$ for each $i \in I$, then $(\cup_{i \in I} U_i) \in \mathcal{T}$.
(3) The intersection of a finite collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$, or equivalently if $\{U_i : i \in I\}$ is a collection such that $U_i \in \mathcal{T}$ for each $i \in I$, then $(\cap_{i \in I} U_i) \in \mathcal{T}$.

Definition 21 (Topological space) *Let X be a non-empty set, and $\mathcal{T}$ a topology for X. The pair $(X, \mathcal{T})$ is called a* topological space*.*

An element x of a topological space X is usually referred as a *point* of X. Whenever it makes no confusion, a topological space $(X, \mathcal{T})$ is denoted by its underlying set X.

Definition 22 (Open set) *Let $(X, \mathcal{T})$ be a topological space. A set $U \in \mathcal{T}$ is called an* open set*.*

Definition 23 (Closed set) *Let X be a topological space. A set $A \subset X$ whose complement A' is open is called a* closed set*.*

Closed sets have the following properties:

(1) $\emptyset$ and X are closed sets.
(2) The intersection of closed sets in X is closed.
(3) Any finite union of closed sets in X is closed.

Example 11 Let X be the set

$$X = \{a, b, c\},$$

and let

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}.$$

It can be verified that $\mathcal{T}$ is a topology on X. The elements $\emptyset$ and X of $\mathcal{T}$ are mutually complementary and both are open sets. The complements of the open sets $\{a\}, \{b\}, \{a, b\}$ are $\{b, c\}, \{a, c\}, \{c\}$ respectively. By definition, these are closed sets of X.

The following two definitions serve as examples of topological spaces.

Definition 24 (Usual topology of a metric space) *Let X be a metric space, and let $\mathcal{T}$ be the collection of all subsets of X which are open sets in the sense of metric spaces. The set $\mathcal{T}$ defines a topology on X and it is called the* usual topology *on X.*

Example 12 The usual topology on the n-dimensional Euclidean space is given by the sets which are open according to the Euclidean distance.

Definition 25 (Discrete topology) *Let X be a non-empty set, and let $\mathcal{T}$ be the collection of all subsets of X. The collection $\mathcal{T}$ is a topology and it is called the* discrete topology *on X, and the topological space $(X, \mathcal{T})$ is called a* discrete space.

Example 13 Let X be a non-empty, and let

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

be a metric for X. This metric space induces a discrete topology on X. In contrast, the collection $\{\emptyset, X\}$ also defines a topology on X.

Definition 26 (Relative subspace) *Let $(X, \mathcal{T}_X)$ be a topological space, and let $Y \subset X$ be a non-empty set. The* relative topology *$\mathcal{T}_Y$ on Y is defined by*

$$\mathcal{T}_Y = \{G = Y \cap U : U \in \mathcal{T}_X\}.$$

The topological space $(Y, \mathcal{T}_Y)$ is called a subspace *of X.*

Example 14 Suppose the topological space $[0, 1]$ defined as a subspace of $\mathbb{R}$. In this space, the interval $[0, \frac{1}{2})$ is an open set.

Definition 27 (Homeomorphism) *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$. f is called an* open mapping *if*

$$(\forall G \in \mathcal{T}_X)(f(G) \in \mathcal{T}_Y),$$

and f is called a continuous mapping *if*

$$(\forall H \in \mathcal{T}_Y)(f^{-1}(H) \in \mathcal{T}_X).$$

The mapping f is called a homeomorphism *if*

(1) f is a bijection,
(2) f is an open mapping and
(3) f is a continuous mapping.

or equivalently

1. f is a bijection,
2'. f is a continuous mapping and
3'. f^{-1} is a continuous mapping.

A function f of a set A is defined by $f(A) = \{y : \exists x \in A \text{ with } f(x) = y\}$. The inverse function $f^{-1}(A)$ is defined by $f^{-1}(A) = \{x \in B : f(x) \in A\}$.

Definition 28 (Homeomorphic) *Let X, Y be topological spaces. X and Y are said to be* homeomorphic *if there exists a homeomorphism from X to Y.*

Definition 29 (Topological property) *Let X be a topological space. Any property of X is said to be a* topological property *if it is possessed by every Y homeomorphic to X.*

For instance, if X is compact and Y is homeomorphic to X, then Y is compact as well. The properties of being connected or Hausdorff are also topological properties. Those properties will be defined later in this section.

Definition 30 (Closure in topological spaces) *Let X be a topological space, and $A \subset X$. The closure of A, denoted by $\mathrm{Cl}(A)$ or $\bar{A}$, is defined by $\mathrm{Cl}(A) = \bigcap_i G_i$, where $A \subset G_i$ and G_i is a closed set of X.*

Definition 31 (Interior in topological spaces) *Let X be a topological space, and $A \subset X$. The interior of A, denoted by $\mathrm{Int}(A)$, is the open set defined by $\mathrm{Int}(A) = \bigcup_i G_i$, where $G_i \subset A$ and G_i is an open set of X. Any point $x \in \mathrm{Int}(A)$ is called an* interior point *of A.*

Definition 32 (Boundary in topological spaces) *Let X be a topological space, and $A \subset X$. The boundary of A, denoted by $\mathrm{Bd}(A)$, is the closed set defined by $\mathrm{Bd}(A) = \mathrm{Cl}(A) \cap \mathrm{Cl}(A')$. Any point $x \in \mathrm{Bd}(A)$ is called a* boundary point *of A.*

In figure 1, the interior, closure and boundary of a subset of $\mathbb{R}^2$ under its usual topology as a metric space are displayed.

Definition 33 (Open cover) *Let X be a topological space. A collection*

$$\{G_i : G_i \text{ is an open set of } X\}$$

is called an open cover *if*

$$\bigcup_i G_i = X$$

Example 15 The set $\{(0, 1/n) : n \in \mathbb{Z}^+\}$ is an open cover for the interval $(0, 1)$ as a subspace of $\mathbb{R}$.

Definition 34 (Subcover) *Let C be an open cover of X. A sub-collection $S \subset C$ is called a* subcover *if it is also an open cover.*

Definition 35 (Compact space) *Let X be a topological space. X is called a* compact space *if every open cover of X has a* finite *subcover.*

Roughly speaking, a compact space X is a topological space for which any collection of open subsets of X whose union is X has a *finite* sub-collection whose union is also X.

For instance, every closed interval of the real line is compact. This fact is known as the *Heine-Borel* theorem. Furthermore, a subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition 36 (Neighborhood of a point) *Let X be a topological space, and $x \in X$. Any open set $U \subset X$ containing x is called a* neighborhood *of the point x.*

Definition 37 (Open base) *Let X be a topological space. An* open base *for X is a collection of open sets such that every open set of X can be expressed as the union of sets in this collection. Equivalently, an open base is a collection of open sets of X such that for every open set G containing a point x there exists a set U in the open base such that $x \in U$ and $U \subset G$.*

The sets in an open base are referred as *basic open sets.* The fact that $\mathcal{B}$ is an open base for a topological space X is expressed by saying that X is *generated* by $\mathcal{B}$.

Example 16 In a metric space the set of all open balls is an open base for the space.

Definition 38 (Second countable space) *Let X be a topological space. X is a* second countable space *if it has a countable open base.*

Example 17 The real line $\mathbb{R}$ has a countable open base given by the set of all open intervals (a, b) with rational end points.

Definition 39 (Hausdorff space) *A* Hausdorff *(or* T_2-space*) is a topological space in which each pair of distinct points have disjoint neighborhoods.*

Example 18 The set $X = \{a, b, c\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ (see example 11) is an example of a topological space which is not Hausdorff since the points a and c have no disjoint neighborhoods.

Example 19 All metric spaces with the usual topology constitute examples of topological spaces which are Hausdorff.

Definition 40 (Connected space) *A* connected space *is a topological space which cannot be expressed as the union of two disjoint non-empty open sets.*

Example 20 Every interval in $\mathbb{R}$ as a subspace of $\mathbb{R}$, and the n-dimensional Euclidean space are examples of connected spaces.

Definition 41 (Connected subspace) *A* connected subspace *of X is a subspace of X which is itself connected.*

Definition 42 (Component) *Let X be a topological space. A* component *of X is a connected subspace which is not properly contained in any other connected subspace of X.*

For instance, every connected space has a single component which is the space itself. In the other hand, in every discrete space each point is a component.

Example 21 Let Y denote the subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$. The subspace Y is not connected, and the sets $[-1, 0)$ and $(0, 1]$ are its components.

Definition 43 (n-manifold) *Let $n \geq 0$ be an integer. An* n-manifold *(or* manifold of dimension n*) is a second-countable Hausdorff topological space where each point has a neighborhood homeomorphic to $\mathbb{R}^n$.*

Definition 44 (n-manifold with boundary) *Let $n \geq 0$ be an integer. An* n-manifold with boundary *is a second-countable Hausdorff topological space where each point has a neighborhood homeomorphic to either $\mathbb{R}^n$ or to the closed upper half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ (by convention $\mathbb{R}^0_+ = \mathbb{R}^0$). The set of all points in an n-manifold with boundary M, having a neighborhood homeomorphic to the closed upper half-space $\mathbb{R}^n_+$ is well defined and it is called the* boundary *of M. It is usually denoted by ∂M.*

It is easy to see that the boundary of a n-manifold with boundary is an $(n-1)$-manifold without boundary. Notice that an n-manifold is just an n-manifold with boundary whose boundary is empty.

Definition 45 (Open manifold) *An* open manifold *is a non-compact manifold without boundary.*

Definition 46 (Closed manifold) *A* closed manifold *is a compact manifold without boundary.*

The following topological spaces are examples of manifolds.

Example 22 Any countable discrete topological space is a 0-manifold.

Example 23 Let $n \geq 1$ be an integer. The subspace $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ of $\mathbb{R}^n$ is an $(n-1)$-manifold.

Example 24 Let $n \geq 1$ be an integer. The subspace $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ of $\mathbb{R}^n$ is an n-manifold with boundary. It can be seen that $\partial B^n = S^{n-1}$.

Example 25 Let $n \geq 1$ be an integer. The subspace of $\mathbb{R}^n$

$$H^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1 \text{ and } x_1 \geq 0\}$$

is an $(n-1)$-manifold with boundary. It can be seen that

$$\partial H^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1 \text{ and } x_1 = 0\},$$

and that this subspace is homeomorphic to S^{n-2}.

Example 26 Let $n \geq 2$ be an integer. The subspace of $\mathbb{R}^n$

$$Q^{n-1} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\| = 1, x_1 \geq 0 \text{ and } x_2 \geq 0\}$$

is an $(n-1)$-manifold with boundary. It is easy to see that

$$\partial Q^{n-1} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\| = 1 \text{ and } x_1 \cdot x_2 = 0)\}.$$

Example 27 Let $a_1 = (1,0,0)$, $a_2 = (0,1,0)$ and $a_3 = (0,0,1)$. The subspace of $\mathbb{R}^3$

$$T = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 : \lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 1\}$$

is a 2-manifold with boundary. It can be seen that

$$\partial T = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 : \lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 1 \text{ and } \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0\}.$$

5 PL-category (Piecewise-linear category)

In order to define the building blocks of PL-objects (piecewise-linear objects) the following technical condition are required.

Definition 47 (Affine independence) *Let $A = \{a_0, a_1, \ldots, a_n\}$ be a set of $n+1$ points in $\mathbb{R}^N$. A is said to be* affine independent *or* geometrically independent *if it does not exist a affine hyperplane of dimension $n-1$ containing all the points in A.*

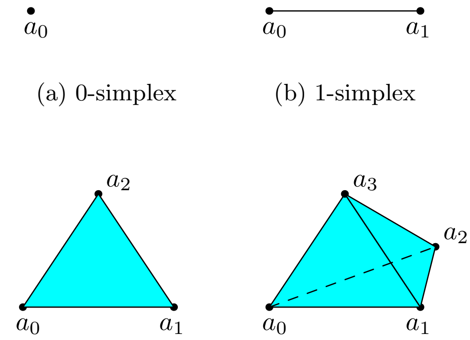
Definition 48 (Simplex) *Let $A = \{a_0, \ldots, a_n\}$ be a set of affine independent points in $\mathbb{R}^N$. The* n-dimensional geometric simplex *or* n-simplex *σ spanned by A is the set of all points $x \in \mathbb{R}^N$ such that*

$$x = \sum_{i=0}^{n} \lambda_i a_i, \quad \text{where} \quad \sum_{i=0}^{n} \lambda_i = 1$$

and $\lambda_i \geq 0$ for $i \in \{0, \ldots, n\}$. The set of reals λ_i are called the barycentric coordinates *of x.*

The simplex σ spanned by $\{a_0, \ldots, a_n\}$ it is denoted by $\sigma = \langle\{a_0, \ldots, a_n\}\rangle$.

As displayed on figure 2, 0-simplices are points, 1-simplices are segments, 2-simplices are triangular regions and 3-simplices are solid tetrahedra.



(a) 0-simplex (b) 1-simplex

(c) 2-simplex (d) 3-simplex

Fig. 2. Examples of simplices in $\mathbb{R}^3$

Every simplex σ in $\mathbb{R}^N$ satisfies the following properties:

(1) σ is a convex set.
(2) σ is a compact set in $\mathbb{R}^N$, i.e. the line segment in $\mathbb{R}^N$ connecting any pair of points of σ lies in σ.
(3) There is one and only one affine independent set of points in $\mathbb{R}^N$ spanning σ.

Definition 49 (Vertex) *Let σ be a n-simplex in $\mathbb{R}^N$. The points $a_0, a_1, \ldots, a_n$ spanning σ are called the* vertices *of σ.*

Definition 50 (Face) *Let σ be a n-simplex in $\mathbb{R}^N$ spanned by $\{a_0, a_1, \ldots, a_n\}$. Any simplex spanned by a subset of $\{a_0, a_1, \ldots, a_n\}$ is called a* face *of σ.*

Example 28 Let $\sigma = \langle\{a_0, a_1, a_2\}\rangle$ be a 2-simplex in some $\mathbb{R}^N$. The faces of σ are σ itself, the 1-simplices $\langle\{a_0, a_1\}\rangle$, $\langle\{a_1, a_2\}\rangle$, $\langle\{a_0, a_2\}\rangle$ and the 0-simplices $\langle\{a_0\}\rangle$, $\langle\{a_1\}\rangle$, $\langle\{a_2\}\rangle$.

Definition 51 (Proper face) *Let σ be a n-simplex in $\mathbb{R}^N$. The faces of σ other than σ itself are called the* proper faces *of σ.*

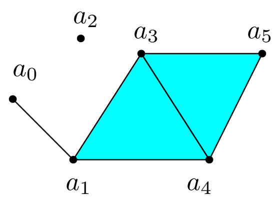
Definition 52 (Boundary of a simplex) *Let σ be a n-simplex in $\mathbb{R}^N$. The* boundary *of σ, denoted by $\mathrm{Bd}(\sigma)$, is the union of all the proper faces of σ.*

Definition 53 (Interior of a simplex) *Let σ be a n-simplex in $\mathbb{R}^N$. The* interior *of σ, denoted by $\mathrm{Int}(\sigma)$, is the set defined by $\mathrm{Int}(\sigma) = \sigma - \mathrm{Bd}(\sigma)$. The set $\mathrm{Int}(\sigma)$ is called an* open simplex.

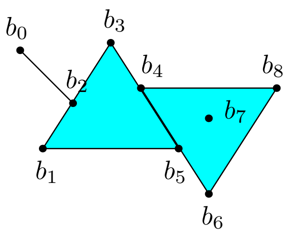
Definition 54 (Properly joined) *Two simplices σ_1, σ_2 are* properly joined *in $\mathbb{R}^N$ if either $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2$ is a (not necessarily proper) face of both.*

Figure 3 shows examples of properly and not properly joined simplices in $\mathbb{R}^2$. In figure 3(a), the 2-simplices $\langle\{a_1, a_4, a_3\}\rangle$ and $\langle\{a_3, a_4, a_5\}\rangle$ intersect each other in the 1-simplex $\langle\{a_3, a_4\}\rangle$ which is a face of both. The simplices $\langle\{a_1, a_4, a_3\}\rangle$ and $\langle\{a_0, a_1\}\rangle$ intersect each other in the 0-simplex $\langle\{a_0\}\rangle$ which is also a face of both. Therefore, this set of simplices is pairwise properly joined.

On the other hand, figure 3(b) displays a set of simplices which is not pairwise properly joined. For instance, the 2-simplices $\langle\{b_1, b_5, b_3\}\rangle$ and $\langle\{b_6, b_8, b_4\}\rangle$ intersect each other in the 1-simplex $\langle\{b_4, b_5\}\rangle$, which is not a face of any of them. Also, $\langle\{b_1, b_5, b_3\}\rangle$ intersects $\langle\{b_0, b_2\}\rangle$ in $\langle\{b_2\}\rangle$, which is not a face of either simplex. Finally, $\langle\{b_7\}\rangle$ intersects (and it is actually contained in) $\langle\{b_6, b_4, b_8\}\rangle$ but it is not a face of the latter.



(a) Properly joined



(b) Not properly joined

Fig. 3. Examples of properly and not properly joined simplices in $\mathbb{R}^2$

Definition 55 (Simplicial complex in $\mathbb{R}^N$) *A simplicial complex K in $\mathbb{R}^N$ is a finite collection of simplices in $\mathbb{R}^N$ such that:*

(1) Every face of an element in K is itself in K.
(2) The elements in K are pairwise properly joined.

Definition 56 (Dimension of a simplicial complex) *Let K be a simplicial complex in $\mathbb{R}^N$. The dimension of K is the largest positive integer r such that K has an r-simplex.*

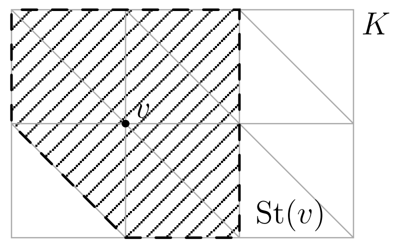
Definition 57 (Subcomplex) *Let K be a simplicial complex in $\mathbb{R}^N$. A* subcomplex *of K is a subset of K which is also a simplicial complex.*

Definition 58 (p-skeleton) *Let K be a simplicial complex in $\mathbb{R}^N$. The p-*skeleton *of K, denoted by $K^{(p)}$, is the subcomplex of K formed by all the simplices in K of dimension at most p. The points in $K^{(0)}$ are called the* vertices *of K.*

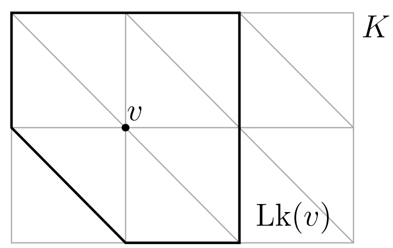
Definition 59 (Star) *Let K be a simplicial complex in $\mathbb{R}^N$. If v is a vertex of K, the* star *of v in K, denoted by $\mathrm{St}(v)$, is the union of the interior of the simplices in K that have v as a vertex. The closure of $\mathrm{St}(v)$ as a subset of $\mathbb{R}^N$, denoted by $\overline{\mathrm{St}}(v)$, is called the* closed star *of v in K.*

Definition 60 (Link) *Let K be a simplicial complex in $\mathbb{R}^N$, and v a vertex of K. The set $\overline{\mathrm{St}}(v) - \mathrm{St}(v)$, denoted by $\mathrm{Lk}(v)$, is called the* link *of v in K.*

In figure 4, a simplicial complex K in $\mathbb{R}^2$ is displayed, where the star and link for a vertex v of K are marked.



(a) Star of v in K



(b) Link of v in K

Fig. 4. Star and link of a vertex v of a simplicial complex K in $\mathbb{R}^2$

Definition 61 (Underlying space or polytope) *Let K be a simplicial complex in $\mathbb{R}^N$. The point set union of the simplices of K, denoted by $|K|$, together with its usual topology as a subspace of $\mathbb{R}^N$, is called the* underlying space *or* polytope *of K.*

The underlying space $|K|$ of a simplicial complex K in $\mathbb{R}^N$ has the following properties:

(1) $|K|$ is a closed and bounded subset of $\mathbb{R}^N$, and so $|K|$ is a compact space.

(2) Each point of $|K|$ lies in the interior of exactly one simplex of K.

Definition 62 (Polyhedron) *A subset of $\mathbb{R}^N$ is called a* polyhedron *if it is the polytope of some simplicial complex in $\mathbb{R}^N$.*

Definition 63 (Triangulation) *Let X be a topological space. If there exists a simplicial complex K in some $\mathbb{R}^N$ such that $|K|$ is homeomorphic to X, then X is called a* triangulable space. *A pair (K, h), where K is a simplicial complex some $\mathbb{R}^N$ and $h : |K| \to X$ is a homeomorphism, is said to be a* triangulation *of X.*

In order to define the notions of orientation of a simplex and oriented simplex the following concepts are required.

Definition 64 (Symmetric group) *Let J_{n+1} denote the set formed by the integers $\{0, \dots, n\}$. A* permutation *of J_{n+1} is a bijection from J_{n+1} onto itself. The set of all permutations of J_{n+1} is a group under the operation of composition. This group is called the* symmetric group in $n+1$ symbols *and it is denoted by $\mathcal{S}_{n+1}$. A* transposition *is an element of $\mathcal{S}_{n+1}$ which is not the identity map but restricts to the identity map in some subset of J_{n+1} having $n-1$ elements.*

There are two facts about symmetric groups which will be useful in defining the notion of orientation:

(1) Any element in a symmetric group can be factored (in a non-unique way) as a product of transpositions.
(2) The parity of the number of factors in any two factorizations in transpositions of a fixed element in a symmetric group is the same.

Let $\sigma = \langle\{a_0, \dots, a_n\}\rangle$ be an n-simplex in $\mathbb{R}^N$. Consider the set

$$\{(a_{s(0)}, \dots, a_{s(n)}) : s \in \mathcal{S}_{n+1}\}.$$

Two elements $(a_{s_1(0)}, \dots, a_{s_1(n)})$, $(a_{s_2(0)}, \dots, a_{s_2(n)})$ are declared equivalent if $s_1 \circ s_2^{-1}$ factors in an even number of transpositions. This is an equivalence relation which determines exactly two equivalence classes. The equivalence class of $(a_{s(0)}, \dots, a_{s(n)})$ will be denoted by

$$\langle a_{s(0)} \dots a_{s(n)} \rangle.$$

It is immediate from the definition that $\langle a_{i_0} \dots a_{i_n} \rangle = \langle a_{j_0} \dots a_{j_n} \rangle$ if and only if any sequence of transpositions taking $(a_{i_0}, \dots, a_{i_n})$ to $(a_{j_0}, \dots, a_{j_n})$ has an even number of factors.

Definition 65 (Oriented n-simplex) *Let $\sigma = \langle\{a_0, \dots, a_n\}\rangle$ be an n-simplex in $\mathbb{R}^N$. Any of the two equivalence classes defined above is called an* orientation *of σ. An* oriented n-simplex *is a simplex with a choice of one of the two possible orientations of it. The oriented simplex σ together with the orientation $\langle a_{i_0} \dots a_{i_n} \rangle$ will be simply denoted by $\langle a_{i_0} \dots a_{i_n} \rangle$.*

Let σ_1, σ_2 be two oriented simplicial complexes in $\mathbb{R}^N$. The equation $\sigma_1 = -\sigma_2$ means that they are equal as unoriented simplices, but carry different orientations.

Example 29 Let $\sigma = \langle\{a_0, a_1, a_2\}\rangle$ be a 2-simplex (see figure 2(c)). The oriented simplices $\langle a_0a_1a_2 \rangle$, $\langle a_1a_2a_0 \rangle$, $\langle a_2a_0a_1 \rangle$ are equivalent and denote one orientation of σ, and the oriented simplices $\langle a_0a_2a_1 \rangle$, $\langle a_1a_0a_2 \rangle$, $\langle a_2a_1a_0 \rangle$ are also equivalent and represent the other orientation of σ.

Definition 66 (Induced orientation) *Let σ be the oriented n-simplex $\langle a_0, \dots, a_n \rangle$ and let τ be the boundary $(n-1)$-simplex $\langle\{a_0, \dots, \hat{a}_i, \dots, a_n\}\rangle$, where $\hat{}$ means deletion of the symbol under it. The oriented $(n-1)$-simplex $(-1)^i \langle a_0 \dots \hat{a}_i \dots a_n \rangle$ is said to carry the orientation* induced *by σ.*

Example 30 Given the oriented simplex $\sigma = \langle abc \rangle$, the induced orientation of σ on its $(n-1)$-simplices are $\langle ab \rangle$, $\langle bc \rangle$ and $\langle ca \rangle$.

Definition 67 (Coherent orientation) *Let σ_1, σ_2 be oriented n-simplices in $\mathbb{R}^N$ such that $\sigma_1 \cap \sigma_2$ is an $(n-1)$-simplex that is face of each of them. It is said that σ_1, σ_2 are* coherently oriented *if they induce opposite orientations on their common $(n-1)$-simplex.*

The most general kind of PL-objects amenable to the notion of an orientation are the pseudomanifolds.

Definition 68 (n-pseudomanifold) *An n-pseudomanifold is a simplicial complex K with the following properties:*

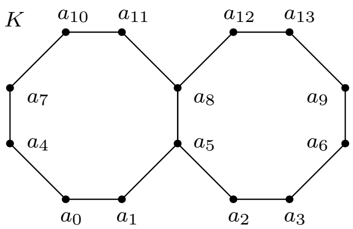
(1) Each simplex in K is a face of some n-simplex in K.
(2) Each $(n-1)$-simplex in K is face of exactly two n-simplices in K.
(3) Given a pair σ_1, σ_2 of n-simplices in K, there exists a sequence of n-simplices beginning at σ_1 and ending at σ_2 such that any two successive terms of the sequence have a common $(n-1)$-face.

Note that by relaxing the second condition in the definition of an n-pseudomanifold, by allowing an $(n-1)$-simplex in K to be face of exactly one or exactly two n-simplices in K, a notion of *n-pseudomanifold with boundary* is obtained.

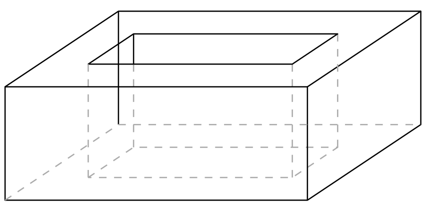
The relationship between n-manifolds (topological spaces) and n-pseudo-manifolds (simplicial complexes) is stated as follows: If X is a triangulable n-manifold then each triangulation K of X is an n-pseudomanifold.

Example 31 The simplicial complex K in $\mathbb{R}^2$ displayed on figure 5(a) is not a 1-pseudomanifold since the 0-simplices $\langle\{a_5\}\rangle$, $\langle\{a_8\}\rangle$ are face of three 1-simplices in K.

Example 32 In figure 5(b), the polytope some simplicial complex K is displayed, which is a triangulation of the torus. Therefore, K is a 2-pseudomanifold.



(a) Example of a simplicial complex K in $\mathbb{R}^2$ which is not an 1-pseudomanifold



(b) Polytope of a 2-pseudomanifold in $\mathbb{R}^3$

Fig. 5. Examples of pseudomanifolds and non-pseudomanifolds

Definition 69 (Orientable n-pseudomanifold) *Let K be an n-pseudomanifold. If there is a way to orient each n-simplex in K such that any two n-simplices having nonempty intersection in K are coherently oriented, K is said to be* orientable*. In this case an* orientation *of K is a particular choice of orientations for the n-simplices in K which is pairwise coherently oriented.*

Example 33 Triangulations of the *Möbius band* of non-orientable simplicial complexes (see figure 6). It an be seen that it is not possible to orient the simplices in such a way they are pairwise coherently oriented.

Definition 70 (Orientable triangulation) *Let X be an n-manifold, and K an n-pseudomanifold corresponding to a triangulation for X. The triangulation K is said to be an* orientable triangulation *of X if K is an orientable n-pseudomanifold.*

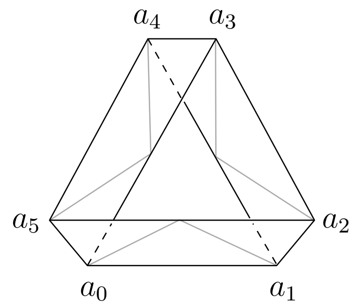
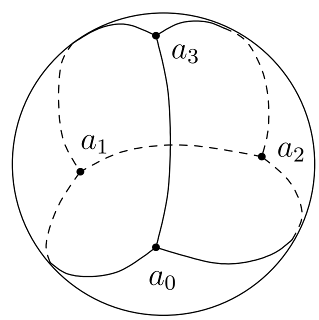


Fig. 6. Example of a non-orientable 2-pseudomanifold in $\mathbb{R}^3$

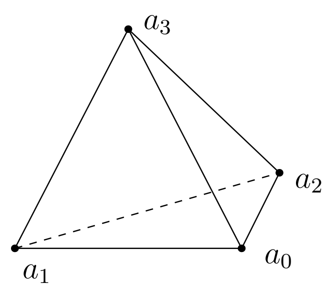
For instance, manifolds of dimension up to three are always triangulable.

Definition 71 (Orientable, oriented n-manifold) *Let X be a triangulable n-manifold. X is said to be orientable if some (and therefore any) triangulation K of X is orientable. Orienting X means specifying a triangulation K of X, together with an orientation.*

Example 34 Some orientable 2-manifolds realized in $\mathbb{R}^3$ are the unitary sphere $S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$ (see figure 7) and the torus $T = \{(x, y, z) : a^2 - z^2 = (\sqrt{x^2 + y^2} - A)^2\}$. An example of a non-orientable 2-manifold is the Möbius band.



(a) 2-manifold S^2



(b) A triangulation of the 2-manifold S^2 in $\mathbb{R}^3$

Fig. 7. Example of a triangulation of the orientable triangulable n-manifold S^2

6 Conclusions

The comprehension of the notions of point-set topology, metric spaces and simplicial complexes is important for people involved in research and development of applications in CAD/CAM/CAE (Computer Aided Design, Manufacturing and Engineering), both at the universities and companies, since such notions constitute an important part of the mathematical background required on problems involving curves, surfaces or solids in the two and three dimensional space.

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