Principal Component Analisis -PCA- and Delone Triangulations for PL approximation C^1-continuous 1-manifolds in $\Re^n$

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Glossary

S	:	Point sample in $\Re^3$.
$C_i(u)$	:	Parametric curve $C_i(u) : \Re \longrightarrow \Re^3$.
$R()$	:	Equivalence relation defined on S.
Π	:	Partition induced by $R()$ on S
Π_i	:	Subset of the partition Π induced by $R()$ on S. $\Pi_i \cap \Pi_j = \cap_{i \neq j}, \quad \cup_i \Pi_i = S$.
m	:	Size (number of points) of the sample.
n	:	Dimension of the sampled space: $n = 3$ in this article.
X	:	Generic Set of m data points with dimension n each.
Σ_x	:	Covariance Matrix for X.
W	:	Rigid Transformation in $\Re^n$.
Y	:	X transformed by W.
ϕ_i	:	Eigenvector of Σ_x.
λ_i	:	Eigenvalue of Σ_x.
P_v	:	Center of Gravity of X.
η	:	Parameter of line $\vec{L}(\eta) = \vec{P_v} + \eta * v$.
T	:	Tape-shaped 2D polygonal region (possibly with holes)
	:	covering S if it is a sample of a planar curve $C_i(u)$.
∂T	:	Boundary of T.
$DT(S)$	:	Delone Triangulation of the point set S.

1 Introduction

The goal of the present article is to present the obtention of a Piecewise Linear (P.L.) approximation for a stochastic point sample $S = \{P_0, P_1, \ldots, P_{N-1}, P_N\}$ from a set of C^1 parametric curves $C_i(u)$ in $\Re^3$, which are 1-manifolds in all but a finite number of points. The $C_i(u))$ curves may be open (1-manifolds with border) or closed (1-manifolds without border) and be in general position in $\Re^3$. However, this work will concentrate on planar or quasi-planar curves as they are likely to include self-intersections (non-manifold neighborhoods). In spite of their possible non-manifold character, the curves are supposed to be sampled following the Shannon or Nyquist criteria.

The proposed methods combines statistical (Principal Component Analysis -P.C.A.) and deterministic techniques (Delone Triangulations) to find P.L. (Piecewise Linear) estimations of the $C_i(u)$.

The process begins with the generation of a partition Π induced on S by an equivalence relation based on the euclidean distance between points of S. The partitions Π_i of Π are calculated with a Transitive Clausure algorithm. Each Π_i then happens to be the sample of a particular curve $C_i(u)$.

For non-planar sets (discussed in other publications) , the P.L. estimations for the $C_i(u)$s may be made by using the Principal Component Analysis (PCA) method alone. The PCA is applied on nearly linear subsets of each Π_i. Therefore, the best (statistical) linear approximation of local neighborhoods of C_i with directed straight segments $\overline{sg}_{i,j}$ is iteratively calculated until the point sample is exhausted.

Our current focus is planar curves, including non-manifold neighborhoods, usually self - intersections of the curve (i.e. "8"-like curves). Even if this non-manifold is sampled according to Nyquist criteria, algorithms for PL approximation are confused, as discussed later. In such cases, it is our goal to obtain either one from two possible results for the PL approximation of $C_i(u)$: (a) two separate coplanar closed polygons and (b) one planar polygon with a thin waist.

As a general remark, point samples of planar curves render quasi-planar point samples. Therefore, perfect planarity in the input data is unlikely.

In another typical situation in surface reconstruction from planar samples a particular level or cross cut k is missing or incomplete. In such cases, point samples from levels $k-1$ and $k+1$ are *borrowed*, and projected onto the (insufficiently sampled) plane k. Then, the cross section of the object lying on plane k is to be recovered from a noisy point set. This point set must be treated with statistical tools (PCA is here used). The cross sections recovered in this manner are the best fit to the planar point cloud contained in plane k.

In such and other cases, algorithms which thread the point set based on Eucliead distance relations usually miss a portion of the points, have termination problems, etc. Those cases cannot be handled with either statistical or deterministic techniques, but a combined approach is required. This is the focus of the present work.

In our present wourk Section 2 examines the related literature reviewed. Section 3 discusses concepts necessary to implement the algorithm and their articulation in reaching the solution. Section 4 presents the results achieved, while section 5 draws the relevant conclusions and propposes bases for future work.

2 Literature Review

The recovery of 1-manifolds in $\Re^3$ from point samples has been approached from the statistical and deterministic points of view. In this article, one seeks: (i) to place the statistical treatment of point samples under the perspective of Principal Component Analysis, given its elegant and consistent underlying theory (applied without modification for 1- and 2-manifold approximation), and (ii) to interleave statistical and deterministic methods (Delone Triangulations and Voronoi Diagrams) to obtain approximations for the 1-manifolds.

2.1 Statistical Approach

The statistical approach for 1-manifold reonstruction from point samples has ancient precursors in Hastie and Stuetzle, 1989 ([6]). In this reference, the authors define *Principal Curves* as smooth ones, which pass through the *middle* of the point cluod. They are also called *self-consistent with*, or *principal curve of*, a cloud of d-dimensional data sample with a probability distribution (μ, σ).

In [7] an algorithm for approximating a set of unorganized points in $\Re^3$ is proposed. No self-intersections are considered, and linear regressions are perfomed on implicit froms $a*x+b*y*c=0$. The authors of the present article have replicated (see next sections) the procedure followed in [7], but (i) applying the formalism of PCA (which does not appear there), and (ii) expressing the linear approximations in parametric form $L(\lambda)=P_0+\lambda*v$ (as opposed to implicit ones). It was found that the statistical aspect is considerably simplified with PCA applied to PL parametric approximation. However, as stated by the authors of [7], the algorithm has problems, discussed ahead, with self-intersecting curves $C_i(u)$. Because of this shortcoming, a complement of PCA with Voronoi-Delone methods is presented here.

2.1.1 Principal Component Analysis

Given $S=\{p_i|p_i\in\Re^n, 1\leqslant i\leqslant m\}$ a set of m samples in $\Re^n$. Without loss of generality one may assume that

$$\mu_1=\mu_2=....\mu_n=0 \tag{1}$$

meaning that the expected value of the n-dimensional distribution or the p_is is the origin of $\Re^n$. Let Σ be the covariance matrix of the m-size sample, where $\Sigma_{i,j}$ is the covariance of the i_{th} against the j_{th} component of the n-component p_i points.

One is interested in rotating S with a transformation R such that the new set $T=\{q_i|q_i\in\Re^n, 1\leqslant i\leqslant m\}$ of transformed samples $q_i=R*p_i$ presents maximal dispersion in the direction of the first axis of $\Re^n$, the second maximal dispersion in the direction of the second axis, and so on. The method is discussed for $\Re^3$, although its original version was formulated for $\Re^n$. Let the unit vectors X_p, Y_p, Z_p be the directions in which S has the largest (σ_x), second largest (σ_y) and smallest variance (σ_z) respectively. It may be shown that

1. The pairs $(\pm X_p,\sigma_x)$, $(\pm Y_p,\sigma_y)$, and $(\pm Z_p,\sigma_z)$ are eigenvector - eigenvalue pairs of the Σ matrix.

$$\begin{aligned}\Sigma*(\pm X_p)&=(\pm X_p)*\sigma_x\\ \Sigma*(\pm Y_p)&=(\pm Y_p)*\sigma_y\\ \Sigma*(\pm Z_p)&=(\pm Z_p)*\sigma_z\end{aligned} \tag{2}$$

2. $\pm X_p$, $\pm Y_p$, $\pm Z_p$ are mutually orthogonal.

$$X_p\bullet Y_p=X_p\bullet Z_p=Z_p\bullet Y_p=0 \tag{3}$$

3. Since $q_i=R*p_i$, it follows that also $R*[X_p,Y_p,Z_p,O_p]=[X_w,Y_w,Z_w,O_w]$ and therefore:

$$R=\begin{bmatrix} X_p & Y_p & Z_p & O_p\\ 0 & 0 & 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix} X_p^T & 0\\ Y_p^T & 0\\ Z_p^T & 0\\ O_p^T & 1\end{bmatrix} \tag{4}$$

Plane VS. Line Data Processing Using PCA .
$[X_p, Y_p, Z_p O_p]$ may be easily conformed to be a right handed coordinate system since $[X_p, Y_p, Z_p]$ is already an orthonormal matrix. As desired, a parametric line $L(\lambda) = \mu_S + \lambda * X_p$ which crosses thorugh "the middle" (μ_S) of the point cloud S is found by sorting and naming the eigenvector-eigenvalue pairs such that $\sigma_x \geq \sigma_y \geq \sigma_z$. For line data, the estimation of the direction vector of the line is the eigenvector X_p, since it is associated to the σ_x eigenvalue representing the maximal dispersion.

Because of facts (2) and (4) it is clear that por cuasi-planar data, the eigenvector Z_p associated to σ_z is the estimation of the direction normal to the plane, since σ_z is by definition the direction of minimal dispersion of the (planar) points.

2.1.2 Point Sample Partition

The point sample S may be originated in the sample of *several* parametric curves $C_i(u)$ in $\Re^3$. Regardless of the method employed to estimate a PL approximation for the curves, it is capital to recognice the fact that the data set must be partitioned into the data sets originated from the individual curves. For such a partition let us define an equivalence relation on the S point set:

Extended Neighborhood. An equivalence relation among points in S is defined, which considers equivalent all points sampled from the same curve $C_i(u)$, whenever the sampling conditions are anisotropic and constant over $\Re^3$. For all $p, q \in S = \{p_0, p_1, p_2, ...p_n\} \subset \Re^3$ one defines the *Extended Neighbor* relation $r()$ as:

$$r(p,q) \Longleftrightarrow (\exists [q_1, q_2, ..., q_w] \mid (q_i \in S) \wedge (q_1 = p) \wedge (q_n = q) \wedge (|q_i - q_{i+1}| \leq \epsilon)) \tag{5}$$

Intuitively, a point p is in the extended neighborhood of a point q , iff there exists a sequence of points of the sample S, starting at p and ending at q such that no two points of the sequence are farther appart than a distance ϵ from each other.

The $r()$ relation defined above is an equivalence relation. It holds that:

$$\begin{aligned} r(P_i, P_i) \\ r(P_i, P_j) \wedge r(P_j, P_k) \rightarrow r(P_i, P_k) \\ r(P_i, P_j) \rightarrow r(P_j, P_i) \end{aligned} \tag{6}$$

The properties of the relation $r()$, Extended Neighbor, in (6) allow to partition the set S into a numer of subsets $S_1, S_2, ..., S_w$ such that $\cup_i S_i = S$ and $S_i \cap S_j = \phi$ for $i \neq j$. Each S_i of the partition is the set of points sampled from the curve $C_i(u)$. The partition of the set S by using the equivalence relation $r()$ is realized by using an algorithm of *transitive clausure*. For details about transitive clausure algorithms or equivalence relations see [11].

3 Methodology

The PL approximation for samples of quasi-planar curves $C_i(u)$ in $\Re^3$ presented here addresses (i) 1-manifolds with or without border (open or closed curves, respectively) and (ii) curves (closed or open) which are 1-manifolds in all but a finite number of points, called self-intersections. It is assumed that two curves $C_i(u)$ and $C_j(u)$ do not intersect each other. If they do so, then we would say that they form *one* non-1-manifold.

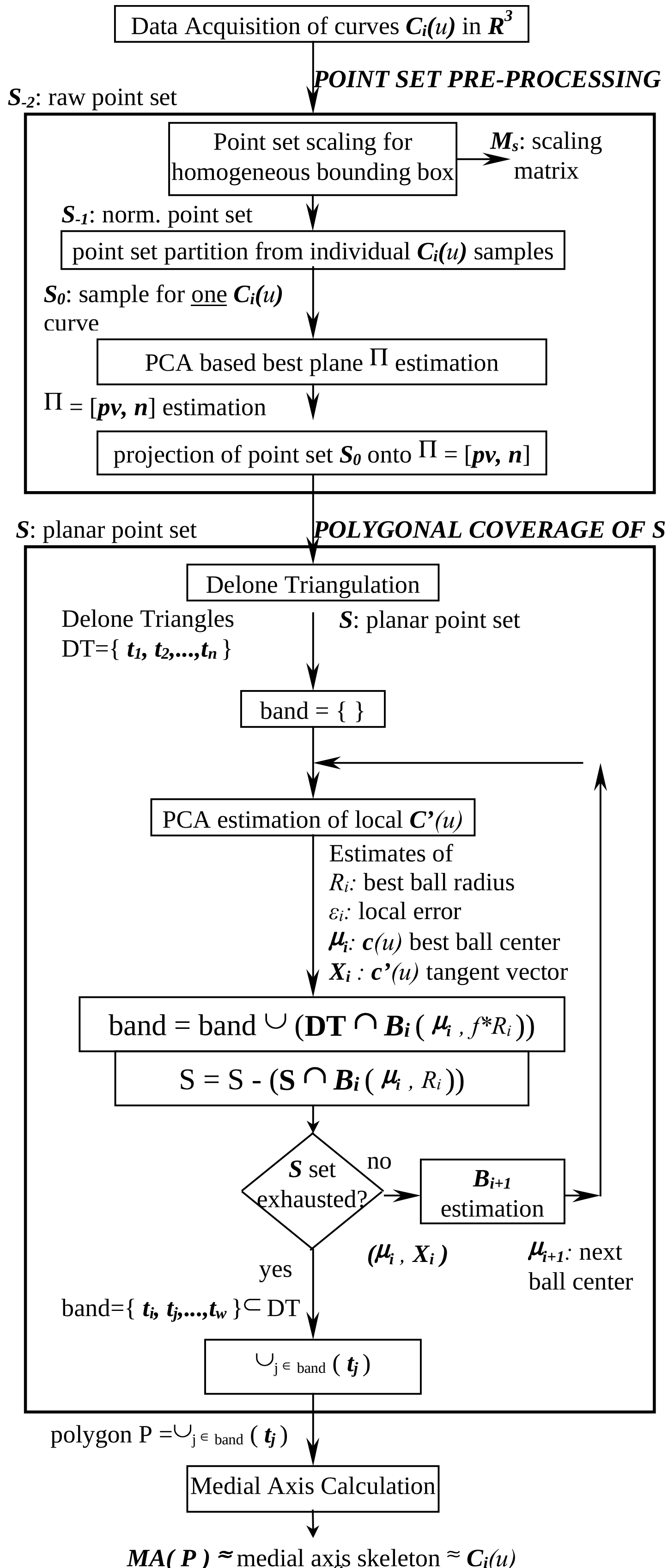
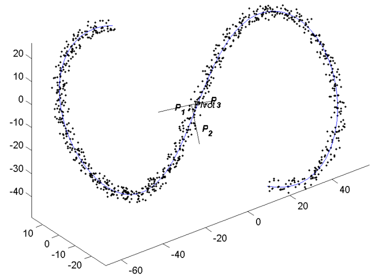
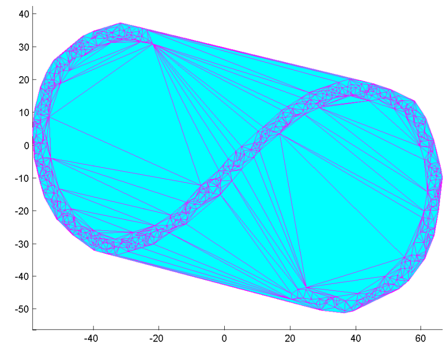


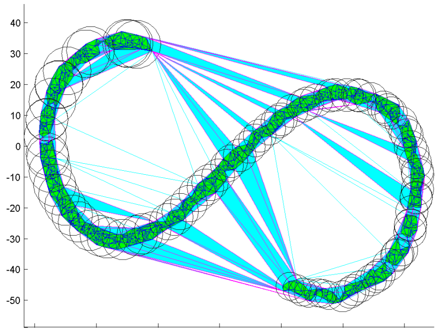
Figure 1: General Layout of PL approximation for quasi-planar samples.



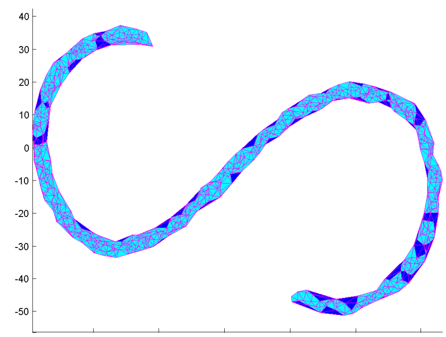
(a) Point Sample of Planar S-shaped $C_i(u)$ Manifold.



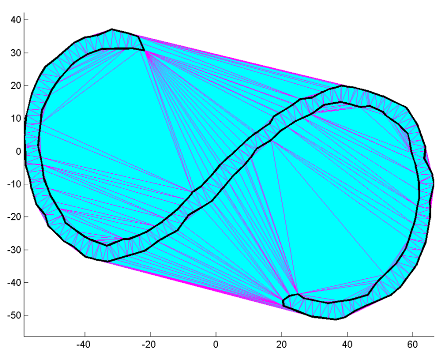
(b) Delone Triangulation of S-shaped Planar Point Sample.



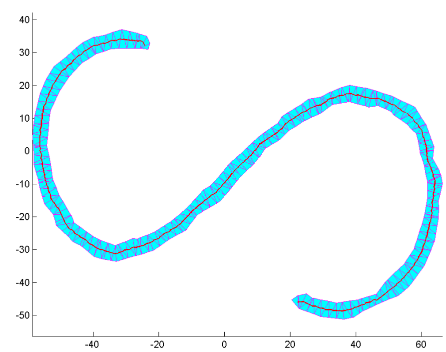
(c) Filtering of Delone Triangulation with PCA Balls



(d) Selected Triangles by Area and Length Criteria



(e) Band Polygon and its Delone Triangulation



(f) Filtered DT and Skeleton

Figure 2: Piecewise Linear Approximation of S-shaped $C_i(u)$ by Combined PCA and Voronoi-Delone Methods

3.1 Data Post- and Pre-Processing

The point data must be pre-processed in this secuence: (i) Scaling: to guarantee that a standard bounding box of the S set is available, since the PCA estimation is sensitive to such aspects. (ii) Partition: to identifiy point sub-sets of S which are originated in the sample of disjoint $C_i(u)$ curves. (iii) Identification of Best Plane: to find a statistically plane Π fit to the cuasi-planar point set S. (iv) Correction to Planar Set: to project S onto Π in order to have a perfectly planar point set. (iv) Transformation of XY Plane: the algorithms presented in this article assume that the data set lies on the plane XY. An underlying assumption is that the results on the plane XY are post-processed to bring them back to the dimension and position inherent to the initial data set. This implies to invert transformations (i) and (iii).

3.2 Principal Curve using PCA

The following sections deal with precise considerations needed to apply the PCA method for an Piecewise Linear (PL) approximation of a $C_i(u)$ parametric curve in $\Re^3$.

3.2.1 Optimal Local Point Set Estimation

Given a point cloud in space π_i, resulting of a statistical sample with variance $[\sigma_x, \sigma_y, \sigma_z]$ from a 1-manifold $C_i(u)$ (possibly with border) in $\Re^3$ one is interested in estimating the tangent line $dC_i(u)/du|_{u=u^*}$, at a point $C_i(u^*)$ of the curve $C_i(u)$. The PCA is applied on points of the sample which are contained inside a ball $B(C_i(u^*), R)$, centered at a seed point P_s with a radius R. Two competing aspects must be compromised: (i) A small enough neighborhood (R) in the data set S must be considered to fit a *linear* estimation of the local tangent. If the ball size R grows too much the data set would include non-linear portions. (ii) With decreasing R the point population around a given center point decreases. The goodness of the linear estimation decreases as a consecuence. To balance (i) and (ii) a search (not discussed here) is conducted for a combination of $C_i(u^*)$ and R, which gives a good linear fitting around a local neighborhood of S determined by the ball $B(C_i(u^*), R)$.

3.3 Principal Curve with PCA and Delone Triangulation

The following discussion will be illustrated using a planar 1-manifold with Border (open $C_i(u)$). Later on, the concepts explained will be applied on self-intersecting planar 1-manifolds.

The quasi-planar point set which is the sample of a planar curve is projected against the best plane fitting the set. This pre-processing is deemed as harmless, since the PL approximation of a planar curve should lie in the best statistical plane fitting the point set. Therefore, in what follows, the point set is considered to be exactly coplanar and confined to the XY plane.

Figure 2(a) shows a data set from a planar non self - intersecting curve sampled with a stochastic process. The figure presents a data set which has been already resized, its best plane estimated, and their points projecte onto this plane, which produces a coplanar subset. The Delone Triangulation of this point set is displayed in Figure 2(b).

The treatment of the point sample with Delone-Voronoi methods stems from the lack of robustness of PCA alone for recovering PL approximations of planar curves from point samples when those curves are nearly or actually self-intersecting. As the PL approximation crosses the first time over the intersection neighborhood, the points in the neighborhood of the intersection are exhausted for purposes of PCA estimation. As the PL curve revisits the intersection neighborhoods no points are available for identifying the trend of curve, and the algorithm tends to look for another point (i.e. curve) neighborhood were to work, withouth really having reproduced the intersection. The result is an incomplete curve stage, therefore missing the self-intersection detail.

Because of this reason, it was decided to determine a minimal polygon T covering S, the point sample of $C_i(u)$. T will resemble a tape (see Figure 2(e)), and may have holes when covering closed or self-intersecting $C_i(u)$. T is a subset of the Delone Triangulation $DT(S)$ of the point set S (see [3] and [5]). In the present work, the triangle removal from $DT(S)$ covering S is helped by the PCA estimations run on local neighborhoods of the point set S (see section 3.3.1). After T is obtained, an *approximation* of the medial axis of T is found. It is called called here a *skeleton of* T. This skeleton contains (in many cases it *is*) the PL approximation of the $C_i(u)$ curve.

3.3.1 Filtering of Delone Triangles

The Delone Triangles will be filtered based on the following criteria: (a) Area, (b) Approximate enclosement in the PCA neighborhood (optimal ball), (c) Edge Length, (d) Vertex perpendicular departure from local tangent lines to $C_i(u)$ and (e) Aspect Ratio. However, in order to apply such methods "reasonable values" of the area and edge length of Delone triangles belonging to T need to be estimated. For such purpose a PCA is run iteratively on neighborhoods of the data set, thus determining the line $\vec{L}(\lambda) = \vec{P_v} + \lambda * v$ that best approaches the tangent to the $C_i(u)$ curve in that neighborhood. Delone triangles contained within a scaled version of this ball, namely $B(C_i(u^*), f_D * R)$ (with $f_D \approx 1.3$ an enlarging factor) might be considered as "typical" of the ones forming the T, and therefore rendering "typical area" $\bar{A}$ and "typical edge length" $\bar{l}$ values. The criteria to classify a Delone triangle as belonging (or not belonging) to the tape are:

1. **Enclosement**: Accept a Delone triangle DT_i if it is contained within the local PCA ball, that is, if $DT_i \subseteq B(P_v, R_o)$ where $B(P_v, R_o)$ is the best local PCA ball (see Figure 2(c)).

2. **Area and Edge Length**: Reject a Delone triangle DT_i if its Area or maximum Edge Length are too large. That is, if $(Area(DT_i) \geq f_A * \bar{A})$ or if $(E_max \geq f_l * \bar{l})$ respectively, for constants f_A and f_l. Figure 2(d) shows Delone Triangles surviving this criterium.

3.3.2 Polygon Synthesis based on Filtered Delone Triangulation

The approximation of the polygon T after application of criteria 1 and 2 is shown in Figure 2(d). As the Delone triangles DT_k making up T are determined, their edges may have 1 or 2 triangles which are incident to it, with the following characteristics:

1. Edges $e_{i,j}$ in which <u>two</u> surviving Delone triangles DT_i and DT_j are incident are internal to the tape-shaped 2D region.

2. Edges e_i in which only <u>one</u> surviving Delone triangle DT_i is incident constitute the boundary ∂T of the 2D tape-shaped region T. They are either in the outermost loop, or in an internal loop.

The classification of internal vs external edges in a 2-manifold with boundary ([2], [9]) is characteristic of Boundary Representations. Additional information on Boundary Representation may be found in [8].

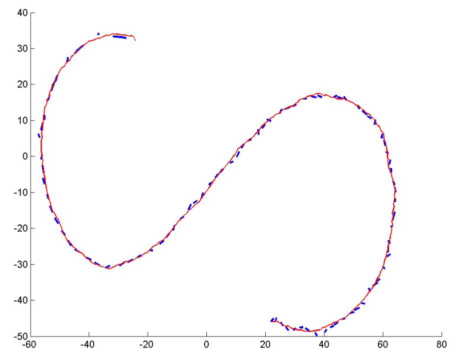


Figure 3: Comparisson of PCA and PCA-VD PL Approximations of S-shaped $C_i(u)$ Manifold.

3.3.3 Medial Axis VS. Principal Curve

Figure 2(d) presents a minimal polygonal region T which covers the point set S. The boundary ∂T of the 2D band-shaped polyon T build by filtering the original Delone Triangulation is marked as black in Figure 2(e). Care must be exercised, however, because such a figure shows a new Delone Triangulation (for the point re-sample of the boundary ∂T). Therefore, the point set for this second Delone Triangulation is not the original point set S.

An approximation to the *medial axis* $MA(T)$ of T is a skeleton $SK(T)$, which is built in the following manner ([4], [1]):

1. Construct the Voronoi Diagram $VD(T)$ and Delone Triangulation $DT(T)$ of the vertices of T (see Figure2(e)).
2. Keep from $DT(T)$ only the Delone triangles contained in T. Call this set $\overline{DT(T)}$.
3. Keep from $VD(T)$ only the Voronoi edges which are finite and are dual to the edges in $\overline{DT(T)}$. Call this set $\overline{VD(T)}$.
4. If $\overline{VD(T)} \nsubseteq T$ then re-sample ∂T with a smaller interval and go to step 1 above. Otherwise, $\overline{VD(T)}$ is the sought skeleton of T, $SK(T)$.

Notice that several re-samples of ∂T may be needed to converge to $SK(T)$. Figure 2(e) shows one of such resamples. The boundary ∂T of the s-shaped polygon T in Figure 2(f) is sampled with a small enough interval. This tight sampling gurantees that the portion of the Voronoi Diagram confined to T, $SK(T)$, is acceptable as an approximation of $MA(T)$, the medial axis of T.

Figure 3 presents a comparisson of the two skeletons: (a) the interrupted one is achieved by direct application of PCA. (b) the continuous skeleton is found by filtering of the Voronoi Diagram assisted by PCA as per the process in Figure 1.

If, as in Figure 2(f), $SK(T)$ is a legal 1-manifold, one may consider it already as a PL approximation of $C_i(u)$. If $SK(T)$ is not a 1-manifold (see Figure 5), it is a super-set of a PL approximation of $C_i(u)$. Removal of the non-manifold neighborhoods would render an acceptable PL curve, approaching $C_i(u)$,

4 Results

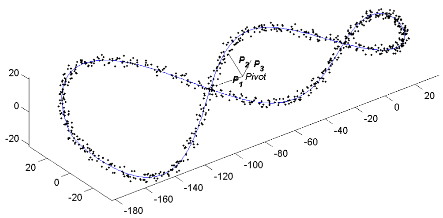
This section illustrates the results of the proposed methodology in dealing with the PL Approximation of planar 1-manifolds without border (closed curves $C_i(u)$). The Figure 4(a) shows the initial point set, along with a coordinate frame attached to the plane Π. As before, the point sample of $C_i(u)$ renders a cuasi-planar point set. According to the discussion, an isotropic scaling was applied to the point set, because PCA is sensitive to dimensional issues. PCA was then applied to estimate the best plane Π fit to the point set, and a modified Householder transformation was used to project all points onto Π (see [10]). In addition, a rigid transformation is used to bring the (now perfectly) planar point set on the plane XY, where the process described in section 3.1 is followed. A planarized point set lying on the XY plane is named "S" hereon. The Delone Triangulation of S is illustrated in the Figure 4(b). An intermediate stage of the screening of Delone triangles included in the PCA balls is shown in the Figure 4(c). Delone triangles contained in the PCA-determined best balls $B(C_i(u), f_D * R)$ are considered part of T. Delone triangles not passing this test may still be included if the Edge Length or Area criteria determine that they may be part of T (see Figure 4(d)). The Edge Lentgh and Area criteria have shown to be very stable and reliable ones, provided that their typical values be found in PCA-derived Ball estimation. After the region TT has been synthezised by clustering Delone triangles chosen according to the PCA, Edge Lentgh and Area criteria, the boundary of the 2-manifold T, ∂T, must be determined. This step is a standard procedure in Boundary Representation construction and is conducted according to rules in section 3.3.2. The *border* ∂T is re-sampled (see Figure 4(e)) and a new Delone Triangulation calculated and purged for this new point set. The purged Delone Triangulation is intended to keep only the Delone Triangles which cover or inclued the point cloud. Triangles from the Delone Triangulation which complete the convex hull of the point set but not inclued points of it are eliminated. In this form, again, the T polygon is re-triangulated, but this time with triangles whose circumscribed center lies inside T. The locii of such centers is $SK(T)$, the skeleton approximation for $MA(T)$, the medial axis of T (see Figure 4(f)).

As seen in Figure 4(f), it is possible that the re-triangulation of T break it into separate regions. This result is a correct one, since indicates the presence of self - intersections in the original set, and corrects it by splitting the tape polygon T into annulus sub-parts T_i. Care must be still exercised, as $SK(T)$ may be outside of a T_i region, as shown in Figure 4(f). This situation, however, is not harmful since the skeletons SK_i do not intersect each other, and therefore serve as PL approximations for the original $C_i(u)$ curve or curves.

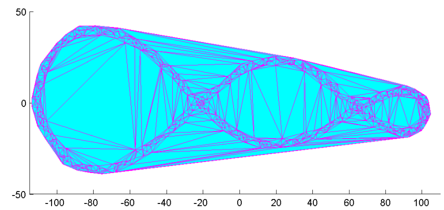
Figures 5(a) and 5(b) show in red the PL approximation by using combined Voronoi-Delone methods plus PCA. Interrupted segments originate in the usage of PCA alone. Both Figures correspond to different point sets scaled in the Y direction, therefore rendering different PL approximations. In both examples, (i) the combined algorithm has superior performance, and (ii) the continuous skeleton approximates very closely the original curve $C_i(u)$.

5 Conclusions and Future Work

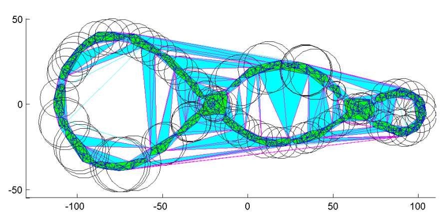
The Piecewise Linear (PL) approximation of open or closed statistically sampled 1-manifolds $C_i(u)$ has been discussed. The proposed method admits a *set* of curves in $\Re^3$ and approximates them by individual PL 1-manifolds. It also admits and correctly splits planar self-intersecting $C_i(u)$ curves into non-self-intersecting PL 1-manifolds. Two methods have been devised and evaluated: (i)



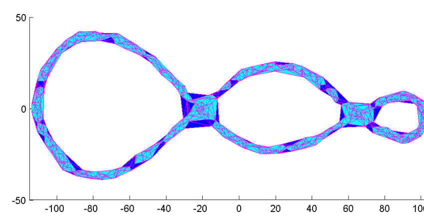
(a) Point Sample of Planar Double-8 $C_i(u)$ Manifold.



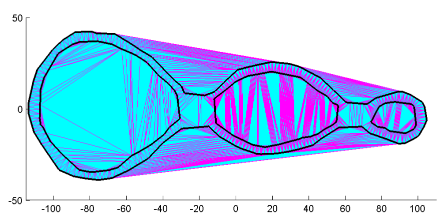
(b) Delone Triangulation of Planar Double-8 Point Sample.



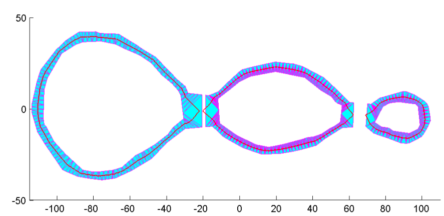
(c) Filtering of Delone Triangulation with PCA Balls



(d) Selected Triangles by Area and Length Criteria

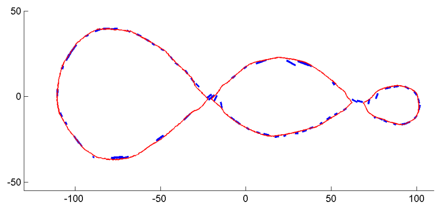


(e) Band Polygon and its Delone Triangulation

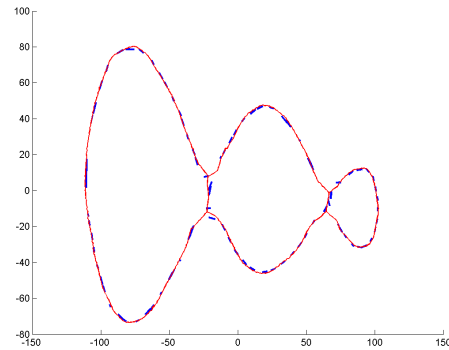


(f) Filtered DT and Skeleton

Figure 4: Process of PL Approximation of Double-8 self-intersecting $C_i(u)$ by Combined PCA and Voronoi-Delone Methods



(a) Result Alternative 1: split Polygons. Comparisson PCA (blue) and VD+PCA (red) Results



(b) Result Alternative 2: joint Polygons. Comparisson PCA (blue) and VD+PCA (red) Results

Figure 5: Final Results. PL Approximations of Double-8 self-intersecting $C_i(u)$ by Combined PCA and Voronoi-Delone Methods

Principal Component Analysis (PCA) march along the point cloud, and (ii) PCA-assisted Voronoi-Delone methods. The second alternative renders significantly better results as the first one. The algorithm presented reaches the point of synthesis of the $SK(T)$ skeleton of the tape-shaped 2D region covering the point set S. This skeleton is a planar graph, but is not necessarily a 1-manifold, as it has branchings. The elimination of the branches is needed, but represents no significant demeanor in the presented results, as existing algorithms for graph splitting are applicable.

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