

# **On the Mathematical Problem that Represents the Steady State Combustion of Solid Fuels**

Javier Schotborgh

Presented in partial fulfillment of the requirements of  
the Magister of Science degree on Applied Mathematics.

Universidad EAFIT  
Facultad de Ciencias y Humanidades  
Maestría en Matemáticas Aplicadas

Adviser: prof. Dr. Carlos A. Cadavid M.  
2007



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# Motivation for the Reader

Nowadays part of the literature on combustion is so specialized that only through long study any significant results can be achieved. On the other hand, another large part of the literature on this subject focuses on developing tools with direct applications in mind. This document fills this gap by first presenting combustion as a chemical process, then explaining how a mathematical model of it is obtained, and finally showing how results in partial differential equations can be applied to understand properties of the combustion process being modelled.



# Chapter 1

## Introduction

### 1.1 Combustion

#### 1.1.1 Definition of Combustion

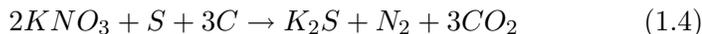
One of the most captivating experiences in childhood is that of contemplating fire. Fire appears as something entirely different from a solid, a liquid or a gas. In particular, this may have lead the Greeks to choose fire (together with earth, air and water) as one of the elements that constitute all other substances. The fire caused by wood burning is just one instance of a number of phenomena referred to collectively as combustion phenomena. A lighted candle, a Bunsen burner, the explosion of gasoline in an internal combustion engine, a flying rocket, and even iron rusting, are instances of combustion processes. With the exception of iron rusting, it is natural to think that these phenomena are somehow variations of the same idea. It was not until the XVIII<sup>th</sup> century that a scientific understanding of what combustion processes essentially are was achieved. In short, a combustion process is an exothermic chemical reaction between a compound (called fuel) and an oxidizer (usually oxygen). Let us consider some examples.

1. One of the first examples of combustion that comes to mind is that of a lighted candle. Candles are made of wax, and wax is a mixture of higher alkanes. An alkane is a molecule formed by a number  $n$  of carbon atoms and  $2n + 2$  hydrogen atoms where  $n \geq 1$  as shown in Figure 1.1. An alkane is said to be a higher alkane if  $n \geq 3$ . The chemical reactions that take place are of the form





4. Very familiar is also the case of the black powder, a solid fuel with chemical equation

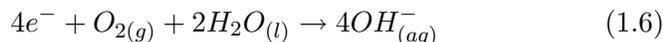


Also, in the following section it will be explained in some detail.

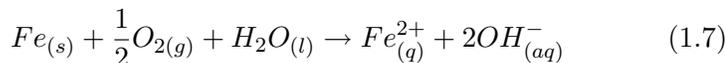
5. There is a type of combustion phenomena, often called *slow combustion*, of which iron rusting is a particularly important example. In this case pure, solid iron oxidizes in water through a number of steps. First, two electrons are taken away from an iron atom due to the presence of  $H^+$  ions



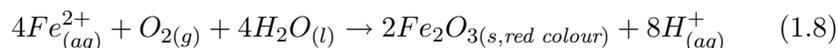
The electrons that are released flow through the iron metal to positively charged regions, where they react with oxygen:



These two half reactions together give the overall reaction:



Experiences with this process, like in car fenders, tend to show that  $Fe^{+2}$  is eventually oxidized further to  $Fe^{+3}$  giving the compound iron(III) oxide (rust):



All examples before are natural manifestation of combustion. Let us see two of them a little further.

### 1.1.2 Examples

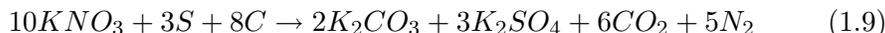
#### Example 1

*Bunsen Burner:* The device safely burns a continuous stream of a flammable gas as natural gas or liquified petroleum gas such as propane, butane, or a mixture of both. The gas flows up through the base through a small hole at the bottom of the barrel and is directed upward. There are slots on the side of the tube bottom to admit air into the stream via the venturi effect.

The amount of air mixed with the gas affects the completeness of the combustion reaction. Less air yields an incomplete and thus cooler reaction, while a gas stream well mixed with air provides oxygen in an equimolar amount and thus a complete and hotter reaction. It is a typical and interesting example of laminar premixed flames with some simplified mathematical models like described in [17].

### Example 2

*Solid fuel (Gunpowder):* A more accurate equation for its reaction is



The products of burning do not follow any simple equation, it generates 55.91 percent solid products: potassium carbonate, potassium sulfate, potassium sulfide, sulfur, potassium nitrate, potassium thiocyanate, carbon, ammonium carbonate. 42.98 percent gaseous products: carbon dioxide, nitrogen, carbon monoxide, hydrogen sulfide, hydrogen, methane, 1.11 percent water. All those products that contain hydrogen are probably contamination, because none of the reactants contains *any* hydrogen. A more complete description of burning of solids is given in [17].

## 1.2 Combustion Physics

As a definition for Combustion, it can be said that is a rapid oxidation generating heat, or both light and heat; also, a slow oxidation accompanied by relatively little heat and no light. Most practical combustion devices belong to the realm of rapid oxidation portion; in this definition, the importance of chemical reactions to combustion, is emphasized. Combustion transforms energy stored in chemical bonds to heat that can be utilized in a variety of ways. Combustion can occur in either a flame or non flame mode, and flames, in turn, are categorized as being either premixed flames or non premixed (diffusion) flames. The two classes of flames, premixed and non-premixed, are related to the state of the reactants, as suggested by their names. In a premixed flame, the fuel and the oxidizer are mixed at the molecular level prior to the occurrence of any significant chemical reaction. The spark-ignition engine is an example where premixed flames occur. Contrarily, in a diffusion flame, the reactants are initially separated, and reaction occurs only at the interface between the fuel and oxidizer, where mixing and reaction both take place. An example of a diffusion flame is a simple candle. In practical devices, both types of flames may be present in

various degrees. Diesel-engine combustion is generally considered to have significant amounts of both premixed and non-premixed or diffusion burning. The term *diffusion* applies strictly to molecular diffusion of chemical species, i.e., fuel molecules diffuse toward the flame from one direction while oxidizer molecules diffuse toward the flame from the opposite direction. In turbulent non-premixed flames, turbulent convection mixes the fuel and air together on a macroscopic basis. Molecular mixing at the same scales, i.e., molecular diffusion, then completes the mixing process so that chemical reaction can take place. It is convenient to refer a flame as a self-sustaining propagation of a localized combustion zone at subsonic velocities; and it is convenient to divide a flame into two zones: the preheat zone, where little heat is released; and the reaction zone, where the bulk of the chemical energy is released. At atmospheric pressure, the flame thickness is quite thin, of the order of millimeters. It is useful to divide the reaction zone further into a region of very fast chemistry followed by a much wider region of slow chemistry. The destruction of the fuel molecules and the creation of many intermediate species occur in the fast-chemistry region. This region is dominated by bimolecular reactions. At atmospheric pressure, the fast-reaction zone is quite thin, typically less than a millimeter. Because this zone is thin, temperature gradients and species concentration gradients are very large. These gradients provide the driving forces that cause the flame to be self-sustaining: the diffusion of heat and radical species from the reaction zone to the preheat zone. In the secondary reaction zone, the chemistry is dominated by three-body radical recombination reactions, and the final burnout of  $CO$  via  $CO + OH \rightarrow CO_2 + H$ . This secondary reaction zone may extend several millimeters in a 1-atm flame. The typical Bunsen-burner flame is a dual flame: a fuel-rich premixed inner flame surrounded by a diffusion flame. The secondary diffusion flame results when the carbon monoxide and hydrogen products from the rich inner flame encounter the ambient air. The shape of the flame is determined by the combined effect of the velocity profile and the heat losses to the tube wall.

Several zones of a candle flame can be seen with the eye. At the bottom is a region that gives off blue light. This light is actually molecular emission from gaseous carbon,  $C_2$ . Further up the flame is a region that is substantially opaque and which gives off yellow light. This is known as the *incandescent region*, and is where hot soot particles glow, giving off light like the filament of a light bulb. The inside part of the flame, near the wick, is oxygen-deficient, and most of the reactions that occur are heat-induced fragmentations and rearrangements. In the outer regions, where oxygen can enter from the surrounding air (oxygen attack), the fragments combine with

oxygen, eventually forming water and carbon dioxide, these concepts are amplified in [8] and [3].

Many factors that affect the burning of a candle. Most of them are of the type that are difficult to vary, such as the air pressure, concentration of oxygen, thermal conductivity of air, and the buoyancy of the hot reaction products. One factor that is easy to vary, however, is wind.

The development to a solid model is inspired on a very useful case as is the fuel used on the STS (Space Transportation System or Space Shuttle) on its SRB (Solid Rocket Booster) that is vital on the initial phase.

### 1.3 Combustion Chemistry

It is known that one of the most important objects relating nature and mankind is fire. Broadly speaking there are two types of fire, flaming and smoldering fires. Flaming fires involve the rapid oxidation of a fuel with associated flame, heat, and light. The flame itself occurs within a region of gas where intense exothermic reactions are taking place. An exothermic reaction is a chemical reaction that takes place within a substance whereby heat and energy are released as the substance change to a simpler chemical form. As chemical reaction occur within the fuel being burned, light is usually emitted as photons are released by the oxidation of the fuel. Depending upon the specific chemical and physical change taking place within the fuel the flame may or may not emit light. The visible flame has no mass; what it is seen as visible flame is actually energy (photons) being released in the form of light by the oxidation of the fuel. The color of the flame is dependent upon the energy level of the photons emitted. Lower energy levels produce colors toward the red end of the light spectrum while higher energy levels produce colors toward the blue end of the spectrum. The hottest flames are white in appearance. A smoldering fire is a flameless form of combustion, deriving its heat from oxidation occurring in the surface of a solid fuel.

A flame is then, the visible (light-emitting) part of a fire. The color and temperature of a flame are dependent of the type of fuel involved in the combustion. When a lighter is held to a candle, the applied heat causes the fuel molecules in the wick to vaporize. In this state they can readily react with the oxygen in the air, which gives off enough heat in the subsequent exothermic reaction to vaporize yet more fuel, thus sustaining a consistent flame. Sufficient energy in the flame will excite the precombusted products, which results in the emission of visible light. As the combustion temperature of a flame increases, so does the average energy of the electromagnetic

radiation given off by the flame. The common distribution of a flame under normal gravity conditions depends on convection, as soot (described in [16]) tends to rise to the top of a flame, making it yellow. In conditions of zero gravity, convection no longer occurs and the flame becomes spherical, with a tendency to become bluer and more efficient. Since combustion problems requiring theoretical analysis are primarily concerned with the flow of reacting and diffusing gases, it must be understood -in addition to chemical thermodynamics- the conservation equations of fluid dynamics, including transport properties and chemical kinetics; I mean, understanding combustion requires a combined knowledge of thermodynamics, heat and mass transfer, and chemical reaction rate theory (chemical kinetics). A review of those aspects are presented on the specialized book as [19], [17] and [13].

Combustion chemistry is very complicated! Many reactions are occurring sequentially and simultaneously. Scientists and engineers do not fully understand the chemistry and mechanics of a candle flame (or other types of flames). The result is the development of models on which the processes are studied, like the model shown in [6].

In general, the fate of the wax molecules is this: the heat of the candle flame first melts the wax, and it rises up the candle wick by capillary action. Farther up the wick, the greater heat vaporizes the wax molecules, which move from the wick into the surrounding space. The heat of the flame and reactive molecules (free radicals) in the flame break apart the wax molecules, in particular stripping hydrogen atoms from the carbon-chain backbone. Some of the carbon chains fragment into gaseous carbon (C<sub>2</sub>) and into small (typically two-carbon atom containing) molecules and molecular fragments. The hydrogen atoms stripped from the wax molecules eventually combine with oxygen atoms from the air to form water molecules. The carbon atoms eventually combine with oxygen to form carbon monoxide and carbon dioxide, but first many of them combine to form very large (as far as molecules are concerned) clumps of carbon-rich solid material, already defined as soot. Some of this soot burns to make carbon dioxide in the candle flame, and sometimes some of it escapes the flame.

As a first step in analyzing the combustion process, we will derive the mathematical model from basic conservation principles.



## Chapter 2

# Mathematical Model

### 2.1 State of the System

Let  $\Omega$  be an open set in  $\mathbb{R}^3$  where a number  $N_s$  of species are moving and reacting with each other according to  $N_r$  chemical reactions. At any particular instant, this system is completely determined by the following collection of real and vector valued functions defined on  $\Omega$ :

1. Temperature  $T : \Omega \times [a, b] \rightarrow \mathbb{R}$ .  $T(x, y, z, t)$  is the temperature at point  $(x, y, z)$  at instant  $t$ .
2. Velocity  $\mathbf{v} : \Omega \times [a, b] \rightarrow \mathbb{R}^3$ .  $\mathbf{v}(x, y, z, t)$  is the velocity vector of a small volume of the mixture of species around the point  $(x, y, z)$  at instant  $t$ .
3. Pressure  $p : \Omega \times [a, b] \rightarrow \mathbb{R}$ .
4. Density  $\rho : \Omega \times [a, b] \rightarrow \mathbb{R}$ .  $\rho(x, y, z, t)$  is the mass of mixture present in a small volume around  $(x, y, z)$  at instant  $t$ , divided by that volume.
5. Specific Enthalpy  $h : \Omega \times [a, b] \rightarrow \mathbb{R}$ .  $h(x, y, z, t)$  is the enthalpy of the mixture contained in a small volume around  $(x, y, z)$  at time  $t$ , divided by the mass of mixture in that volume.
6. Mass fractions  $y_i : \Omega \times [a, b] \rightarrow [0, 1]$ ,  $i = 1, \dots, N_s$ .  $y_i(x, y, z, t)$  is the mass of species  $i$  present in a small volume around  $(x, y, z)$  at instant  $t$ , divided by the mass of mixture present in that same volume. By definition,  $\sum_{i=1}^{N_s} y_i(x, y, z, t) = 1$  for every  $(x, y, z, t)$ .

The minimum set of necessary properties is called the *state* of the system, and it is assumed that the evolution of the flame in time is determined by its state at some initial time  $t = t_0$ , in combination with outside influences, called boundary conditions. In the model used, the state is given by temperature, velocity, pressure and the first  $(N_s - 1)$  mass fractions.

When the state is given, the last mass fraction (for which nitrogen gas, the inert bulk species, is the wisest choice) follows from the notion that the mass fraction should add up to 1, so

$$y_{N_s} = 1 - \sum_{j=1}^{N_s-1} y_j. \quad (2.1)$$

The density  $\rho$  can be derived from the state, using the *equation of state or gas law*. For ideal gases it is given by

$$\rho = \frac{pM}{R_0T}, \quad (2.2)$$

where  $R_0$  is the universal gas constant and  $M$  is the average molecular mass (the reciprocal weighted average of the specific molecular masses  $M_i$  of the species involved)

$$M = \left( \sum_{i=1}^{N_s} \frac{y_i}{M_i} \right)^{-1} \quad (2.3)$$

The pressure dependence of the density can be ignored (laminar combustion), so almost the same results are obtained when the equation of state 2.2 is replaced by

$$\rho = \frac{p_0M}{R_0T}, \quad (2.4)$$

with  $p_0$  the ambient pressure in the flame. This equation of state follows from the so-called *combustion or isobaric* approximation, which is valid when the gas velocity  $\mathbf{v}$  is much smaller than the speed of sound  $c$ . Here the pressure is taken to be a constant, but not in the momentum equations as will be seen. Other property which can be derived from the state is called the specific enthalpy  $h$ , being the sum of the chemical and thermal enthalpy. It will also be treated later in this section.

The Reynolds' Transport Theorem is used to derive the conservation laws for a flowing gas. To formulate the theorem, a *material volume*  $\mathcal{V}(t)$  is considered, which moves through the domain with velocity  $\mathbf{v}(x, y, z, t)$  in such a way, that it contains the same gas at all times. Now let  $b(x, t)$  be the density of a certain property (mass, enthalpy, momentum) per unit mass at

$x$  and at time  $t$ . The total amount  $B$  of this property, contained in  $\mathcal{V}(t)$ , is then equal to

$$B(t) = \int_{\mathcal{V}(t)} \rho b dV. \quad (2.5)$$

The theorem now states that the material derivative of  $B$  is given by

$$\frac{dB}{dt} = \int_{\mathcal{V}(t)} \frac{\partial \rho b}{\partial t} + \oint_{\partial \mathcal{V}(t)} \rho b \mathbf{v} \cdot \mathbf{n} dS, \quad (2.6)$$

with  $\mathbf{n}$  the outward unit vector.

This derivative can also be equated to the difference of the total production inside the volume  $\mathcal{V}(t)$  on one hand and the net transfer out of the volume on the other hand. Therefore, if  $\mathbf{s}(x, y, z, t)$  denotes the production intensity of the property per unit of time per unit volume, and if  $f_b$  is the flux of the property  $b$ , per unit area per unit time, then the time derivative of  $B$  is also equal to:

$$\frac{dB}{dt} = \int_{\mathcal{V}(t)} \mathbf{s} dV - \oint_{\partial \mathcal{V}(t)} f_b \cdot \mathbf{n} dS. \quad (2.7)$$

The two expressions must be identical, so

$$\int_{\mathcal{V}(t)} \frac{\partial \rho b}{\partial t} + \oint_{\partial \mathcal{V}(t)} \rho b \mathbf{v} \cdot \mathbf{n} dS = \int_{\mathcal{V}(t)} \mathbf{s} dV - \oint_{\partial \mathcal{V}(t)} f_b \cdot \mathbf{n} dS. \quad (2.8)$$

Since 2.8 holds for arbitrary  $\mathcal{V}(t)$ , we can apply Gauss' theorem and derive the generic conservation law in the form of a partial differential equation

$$\frac{\partial \rho b}{\partial t} + \nabla \cdot (\rho b \mathbf{v}) = s - \nabla \cdot f_b. \quad (2.9)$$

We can now apply 2.9 to several properties of the gas, to derive the conservation equations which describe the flame's behavior. A detailed general case may also be studied in [4].

### 2.1.1 Conservation of Mass

In a gas flame, mass can never be created or destroyed. Therefore, by setting  $b = 1$  and  $f_p = 0$  in the conservation law 2.9, we obtain the equation describing conservation of *mass*, and find as mentioned in [18]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.10)$$

This can also be expressed for each species  $i$  by

$$\frac{\partial}{\partial t}(\rho y_i) + \nabla \cdot (\rho y_i \vec{v}_i) = \dot{r}_i. \quad (2.11)$$

where  $\dot{r}_i$  is the rate of production (or consumption) for each species  $i$ . As mentioned in [2], in a closed system, it is necessary that  $\sum_{i=1}^N \dot{r}_i = 0$ . The equation for the total conservation of mass is obtained by summing the equations in 2.11, resulting again in

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

### 2.1.2 Conservation of Momentum

Let us set  $\mathbf{b} := \mathbf{v} = (u, v, w)^T$ , so we apply the conservation law to *momentum*. The velocity is a vector field and this increases the notation of the conservation law 2.9. We need to write it is for each of the velocity components. For that purpose, let  $s_j$ , ( $j = x, y, z$ ) denote the production of momentum in the directions  $x, y, z$ , and  $f_j$  the corresponding momentum fluxes. Then we get

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) &= s_x - \nabla \cdot f_x, \\ \frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v \mathbf{v}) &= s_y - \nabla \cdot f_y, \\ \frac{\partial \rho w}{\partial t} + \nabla \cdot (\rho w \mathbf{v}) &= s_z - \nabla \cdot f_z, \end{aligned}$$

which can be abbreviated using the tensor  $\mathbf{F}$ , and the tensor product  $\otimes$ :

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \mathbf{s} - \nabla \cdot \mathbf{F} \quad (2.12)$$

In a gas flame, the cause for the change of momentum are the gravitational force given by  $\rho g$ , which appears in the equation as a source term, and those forces which transfer momentum from one fluid element to another. These forces are pressure forces  $\nabla \cdot F_p$ , and viscous forces  $\nabla \cdot F_v$ . Pressure forces work with equal strength in all directions, and the only form for this type of tensor is given by  $F_p = pI$ . The viscous forces are caused by velocity gradients, and under assumptions of *newtonian behavior* and *isotropy*, are given by

$$F_v = -\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) - \frac{2}{3}I(\nabla \cdot \mathbf{v}). \quad (2.13)$$

with  $\mu$  a scalar named the *dynamic viscosity*. Thus, the *momentum equation* is

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \rho g - \nabla p + \nabla \cdot (\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3}I(\nabla \cdot \mathbf{v}))) \quad (2.14)$$

### 2.1.3 Conservation of Species

An effective approach for a model of transfer of mass of each species between fluid elements, is through the generalized *Law of Fick*, which states that the transfer of mass of a species depends linearly on, and is directed in opposite direction of, the concentration gradient. So, according to [18]

$$f_i = -\rho D \nabla y_i, \quad (2.15)$$

with  $i = 1, \dots, N_s - 1$  and  $D$  the *mixture averaged mass coefficient*. Subsequent substitution into 2.9 leads to

$$\frac{\partial \rho y_i}{\partial t} + \nabla \cdot (\rho \mathbf{v} y_i) = \nabla \cdot (\rho D \nabla y_i) + s_i, \quad (2.16)$$

with  $i = 1, \dots, N_s - 1$  and where  $s_i$  is the source term or *chemical production term*. Equation 2.16 is used to describe the behavior of the first  $(N_s - 1)$  species, as the last one is derived from 2.1. The mass transfer flux of species number  $N_s$  can be derived from other fluxes, because there is no transfer of *total mass* by diffusion, yielding

$$f_{N_s} = - \sum_{i=1}^{N_s-1} f_i. \quad (2.17)$$

### 2.1.4 Conservation of Energy

Since no energy is created or destroyed in a gas flame, is logical to apply the conservation law to the specific energy  $E$  of the gas. When the kinetic energy is very small relative to the internal energy (heat, chemical energy), we may consider only the specific internal energy  $e$ . Energy is also used for the expansion of the gas when the temperature rises, and some inconvenient terms are involved. Instead, it is more convenient to consider the specific *enthalpy*  $h$ , given by

$$h := e + \frac{p}{\rho}. \quad (2.18)$$

The enthalpy is a function of temperature and mass fractions, and takes the form

$$h = \sum_{i=1}^{N_s} y_i h_i. \quad (2.19)$$

where  $h_i^0$  is called the *specific enthalpy* of species  $i$ , and is given by

$$h_i = h_i^0 + \int_{T_0}^T c_{pi}(\tau) d\tau, \quad (2.20)$$

where  $h_i^0$  is called the *specific enthalpy of formation at the reference temperature  $T_0$* , and  $c_{pi}$  is the *specific heat* of the  $i$ -th species.

Enthalpy is not always conserved, because viscosity and changes in the ambient pressure may change it. In a gas flame, these terms are found to be a lot smaller than the chemical and thermal terms, and they can be neglected. Thus the generic conservation law 2.9 can be applied to the enthalpy. To do so, it is needed a model for transfer of enthalpy between fluid elements. The enthalpy transfer mechanisms are, the transfer due to mass diffusion, for which there is a model; and the heat conduction, for which the *Fourrier law* is applied where the heat conduction depends linearly on, and is directed in the opposite direction of the temperature gradient. Then we have

$$f_T = -\lambda \nabla T, \quad (2.21)$$

where  $\lambda$  is the *thermal conductivity* of the gas mixture. The mass diffusion, for which the model is given by 2.15 and 2.17, yields the term

$$f_m = \sum_{i=1}^{N_s} h_i f_i = - \sum_{i=1}^{N_s-1} h_i^* \rho D \nabla y_i. \quad (2.22)$$

where  $h_i^* := h_i - h_{N_s}$  is the *relative enthalpy* of the species number  $i$ . Substitutions into 2.9 leads to the *Enthalpy Equation*

$$\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho \mathbf{v} h) = \nabla \cdot (\lambda \nabla T) + \sum_{i=1}^{N_s-1} \nabla \cdot (h_i^* \rho D \nabla y_i). \quad (2.23)$$

Substitution of 2.19 and 2.16, and application of the chain and product rules for differentiation, produces the *Temperature Equation*

$$c_p \frac{\partial \rho T}{\partial t} + c_p \nabla \cdot (\rho \mathbf{v} T) = \nabla \cdot (\lambda \nabla T) - \sum_{i=1}^{N_s} c_{pi} f_i \cdot \nabla T - \sum_{i=1}^{N_s-1} h_i^* s_i, \quad (2.24)$$

where the *mixture average specific heat*  $c_p$  is given by

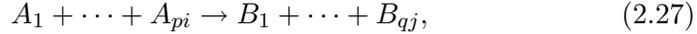
$$c_p := \sum_{i=0}^{N_s} c_{pi} y_i. \quad (2.25)$$

For a given gas flame, the reactions from 1 to  $N_r$  are considered or play a role. The model itself can be of the style of *one step* and of the style *skeletal*. For an arbitrary reaction  $j$ ,  $r_j$  denotes the reaction rate, that is the number of times in moles that the reaction takes place per unit volume per unit time; in this case the chemical production term  $s_i$  is expressed by

$$s_i = M_i \sum_{j=1}^{N_r} \nu_{ij} r_j, \quad (2.26)$$

where  $\nu_{ij}$  is the *stoichiometric coefficient* or the number of molecules of species number  $i$  which are produced every time reaction  $j$  takes place.

After considering a *collision model* the reaction rates are obtained. For the reaction,



it is assumed that the reaction can only take place if a molecule of every species  $A_1$  through  $A_{pj}$  is present in a small volume  $\Delta V$  in which they collide. The probability of this to happen is proportional to the concentration of all the participating species ( $m_{j1}$  through  $m_{jpi}$ ), so

$$r_j = k_j(T) \rho^{pj} \prod_{i=1}^{pj} y_{m_{ji}}, \quad (2.28)$$

where  $k_j(T)$  is a temperature dependent *specific reaction rate coefficient*. As argued by Arrhenius in 1889, an appropriate representation is

$$k_j(T) = B_j e^{-E_j/RT}, \quad (2.29)$$

where the frequency factor  $B_j$  and the activation energy  $E_j$  do not depend on the temperature. When the reactions behavior cannot be represented correctly in that way, it is used a slightly modified form of the reaction, chosen according to

$$k_j(T) = A_j T^{\alpha_j} e^{-E_j/RT}. \quad (2.30)$$

Then, the expression for the chemical production term  $s_i$  is:

$$s_T := - \sum_{i=1}^{N_s-1} h_i^* s_i. \quad (2.31)$$

The model used can be summarized as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.32)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \rho \mathbf{g} - \nabla p + \nabla \cdot (\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3}I(\nabla \cdot \mathbf{v}))) \quad (2.33)$$

$$\frac{\partial \rho y_i}{\partial t} + \nabla \cdot (\rho \mathbf{v} y_i) = \nabla \cdot (\rho D \nabla y_i) + s_i, i = 1, \dots, N_s - 1, \quad (2.34)$$

$$c_p \frac{\partial \rho T}{\partial t} + c_p \nabla \cdot (\rho \mathbf{v} T) = \nabla \cdot (\lambda \nabla T) + \sum_{i=1}^{N_s-1} (c_{pi} - c_p N_s) \rho D \nabla T + s_T. \quad (2.35)$$

These equations, are equivalent to the following set expressed by [2] as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.36)$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \left( \Delta \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right) \quad (2.37)$$

$$\rho \left( \frac{\partial y_i}{\partial t} + \mathbf{v} \cdot \nabla y_i \right) = \nabla \cdot (\rho D \nabla y_i) + m_i (\lambda_i - \nu_i) B_0 e^{-E/RT} \prod_{j=1}^N \left( \frac{\rho y_j}{m_j} \right)^{\nu_j} \quad (2.38)$$

$$\begin{aligned} \rho C_v \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \\ \nabla \cdot (k \nabla T) - p (\nabla \cdot \mathbf{v}) + 2\mu \left[ -\frac{1}{3} (\nabla \cdot \nabla \mathbf{v})^2 + D : \nabla \otimes \mathbf{v} \right] \\ - \sum_{i=1}^N h_i m_i (\lambda_i - \nu_i) B_0 e^{-E/RT} \prod_{j=1}^N \left( \frac{\rho y_j}{m_j} \right)^{\nu_j} \end{aligned} \quad (2.39)$$

$$p = \rho RT. \quad (2.40)$$

## 2.2 Conditions for the Model

The single one-step irreversible reaction that we will consider is of the form:



where  $F$  represents fuel,  $O$  represent oxidant,  $P$  represents the product, and where  $\nu_F$ ,  $\nu_O$ , and  $\lambda_P$  are stoichiometric constants. This reaction involves three mass fractions:  $y_F$ ,  $y_O$ , and  $y_P$ . If both fuel and oxidant are present

in correct proportion, then both are entirely consumed in the process. In this case the initial values  $y_{F_0}$  and  $y_{O_0}$  are of the same order of magnitude, so the reaction rate is strongly dependent on both mass fractions. However, if  $y_{F_0} \gg y_{O_0}$ , then the reaction rate is weakly dependent on  $y_F$  since  $y_F$  does not change much. Since  $y_F$  is approximately constant, we ignore its species equation and consider only the single species equation for  $y_O$ .

Note that the stoichiometric mixture of fuel and oxidant satisfies  $\frac{y_O}{y_F} \doteq \frac{\nu_O}{\nu_F}$ . Choose  $m = \nu_F + \nu_O$ ,  $\lambda_O = 0$ ,  $h = h_O$ ,  $y = y_0$ , and

$$B = B_0 \nu_0 m_0 \left( \frac{\nu_F}{\nu_0} \right)^{\nu_F} m_0^{-\nu_O} m_F^{-\nu_F};$$

then the system becomes

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \rho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) &= -\nabla p + \mu \left[ \Delta \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right] \\ \rho(y_t + \mathbf{v} \cdot \nabla y) &= \nabla \cdot (\rho D \nabla y) - B \rho^m y^m e^{-\frac{E}{RT}} \\ \rho C_v (T_t + \mathbf{v} \cdot \nabla T) &= \\ \nabla \cdot (k \nabla T) - p(\nabla \cdot \mathbf{v}) & \\ + 2\mu [\mathcal{D} : \nabla \otimes \mathbf{v} - \frac{1}{3} (\nabla \cdot \mathbf{v})^2] + \mathcal{B} h \rho^m y^m e^{-\frac{E}{RT}} & \\ p &= \rho R T. \end{aligned}$$

The combustion model just mentioned can be nondimensionalized in a rational manner in order to elucidate the significant parameters. Assume initially that a reactive, viscous, heat conducting, compressible gas is in equilibrium state defined by the dimensional quantities  $p_0 = p(\vec{x}, 0)$ ,  $\rho_0 = \rho(\vec{x}, 0)$ ,  $T_0 = T(\vec{x}, 0)$ ,  $y_0 = y(\vec{x}, 0)$ , and  $\vec{u}_0 = \vec{u}(\vec{x}, 0)$ .

At time  $t = 0$ , a small initial disturbance is created on a length scale  $L$ . Define  $\vec{x} = \vec{x}/L$  as the new position vector. Let  $t_R$  be a reference time (to be determined later). Let's define  $\bar{t} = t/t_R$  as a new time scale. Nondimensionalize the system of variables:  $\bar{p} = p/p_0$ ,  $\bar{T} = T/T_0$ ,  $\bar{y} = y/y_0$ , and  $\underline{\mathbf{v}} = \mathbf{v}/(L/T_R)$ . Also nondimensionalize the quantities:  $\bar{\mu} = \mu/\mu_0$ ,  $\bar{D} = D/D_0$ ,  $\bar{C}_p = C_p/C_{p0}$ ,  $\bar{C}_v = C_v/C_{v0}$ ,  $\bar{k} = k/k_0$ , and  $\bar{K} = K/K_0$ , where  $K = k/(\rho C_p)$  is the thermal diffusivity.

In the scaling of the system it will be used the following quantities:  $\gamma = C_{p0}/C_{v0}$ , the gas parameter;  $\varepsilon = RT_0/E$ , the non dimensional inverse of

the activation energy;  $P_r = C_{p0}\mu_0/k_0$ , the Prandtl number; the Lewis number  $Le = D_0/K_0$ ; the initial sound speed  $C_0 = \sqrt{\gamma RT_0}$ ; the acoustic time scale  $t_A = L/C_0$ ; the conduction time scale  $t_C = L^2/K_0$ ; and the non dimensional heat of reaction  $\bar{h} = h_{y_0}/(C_{v0}T_0)$ . Substituting these into the equations above and dropping the bar notation gives us the non dimensional model

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.41)$$

$$\rho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) = -\frac{1}{\gamma} \left(\frac{t_R}{t_A}\right)^2 \nabla p + P_r \left(\frac{t_R}{t_C}\right) \mu [\Delta \mathbf{v} + \frac{1}{3} \nabla(\nabla \cdot \mathbf{v})] \quad (2.42)$$

$$\rho(y_t + \mathbf{v} \cdot \nabla y) = Le \left(\frac{t_R}{t_C}\right) \nabla \cdot (\rho D \nabla y) - t_R B \rho^m y^m e^{-1/\varepsilon T} \quad (2.43)$$

$$\begin{aligned} \rho C_v (T_t + \mathbf{v} \cdot \nabla T) = & \\ & \gamma \left(\frac{t_R}{t_C}\right) \nabla \cdot (k \nabla T) - (\gamma - 1) p (\nabla \cdot \mathbf{v}) \\ & + 2\mu \gamma (\gamma - 1) P_r \left(\frac{t_A}{t_R t_C}\right) [\mathcal{D} : \nabla \otimes \mathbf{v} - \frac{1}{3} (\nabla \cdot \mathbf{v})^2] \\ & + t_R B h \rho^m y^m e^{-1/\varepsilon T} \end{aligned} \quad (2.44)$$

$$p = \rho T. \quad (2.45)$$

If the single chemical species is a *solid* in a bounded container  $\Omega \subset \mathbb{R}^3$ , then  $\mathbf{v} = 0$ ,  $\rho = 1$ ,  $\gamma = 1$ , and the ratio  $t_R/t_C = \mathcal{O}(1)$ . Thus the last group of equations reduces to the reaction-diffusion system which can be written as

$$\begin{cases} T_t - \Delta T &= \varepsilon \delta y^m \exp\left(\frac{T-1}{\varepsilon T}\right) \\ y_t - \beta \Delta y &= -\varepsilon \delta \Gamma y^m \exp\left(\frac{T-1}{\varepsilon T}\right) \end{cases} \quad (2.46)$$

with  $(x, t) \in \Omega \times (0, \infty)$  and initial-boundary conditions

$$\begin{cases} T(x, 0) = 1, \quad y(x, 0) = 1, \quad x \in \Omega \\ T(x, t) = 1, \quad \frac{\partial y(x, t)}{\partial \eta(x)} = 0, \quad (x, t) \in \partial\Omega \times (0, \infty) \end{cases} \quad (2.47)$$

where  $\beta \geq 0$ ,  $\Gamma > 0$ , and  $\delta > 0$  is the Frank-Kamenetski parameter. For all fuels of interest, the parameter  $\varepsilon$  is assumed small and using the method of

activation energy asymptotics, and letting  $T = 1 + \varepsilon\theta$  and  $y = 1 - \varepsilon c$  be first order approximations, IBVP 2.46 - 2.47 can be written as

$$\begin{cases} \theta_t - \Delta \theta &= \delta(1 - \varepsilon c)^m \exp(\frac{\theta}{1 + \varepsilon\theta}) \\ c_t - \beta \Delta c &= \delta\Gamma(1 - \varepsilon c)^m \exp(\frac{\theta}{1 + \varepsilon\theta}) \end{cases} \quad (2.48)$$

with  $(x, t) \in \Omega \times (0, \infty)$ , and initial-boundary conditions

$$\begin{cases} \theta(x, 0) = 0, c(x, 0) = 0, & x \in \Omega \\ \theta(x, t) = 0, \frac{\partial c(x, t)}{\partial \eta(x)} = 0, & (x, t) \in \partial\Omega \times (0, \infty) \end{cases} \quad (2.49)$$

For  $\varepsilon \ll 1$ , the activation energy method decouples 2.48 and it is only necessary to consider the *solid fuel ignition model*

$$\theta_t - \Delta \theta = \delta e^\theta, \quad (x, t) \in \Omega \times (0, \infty) \quad (2.50)$$

with initial-boundary conditions

$$\begin{cases} \theta(x, 0) = 0, & x \in \Omega \\ \theta(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty) \end{cases} \quad (2.51)$$

and the associated *steady-state model*

$$\begin{cases} -\Delta \psi = \delta e^\psi, & x \in \Omega \\ \psi(x) = 0, & x \in \partial\Omega \end{cases} \quad (2.52)$$



## Chapter 3

# Existence on General Domains

Let  $\gamma$  be a real number in the interval  $(0, 1]$  and let  $\Omega$  be an open, connected and bounded subset of  $\mathbb{R}^n$  such that for each  $x = (x_1, \dots, x_n) \in \partial\Omega$  there exist:

1. an open ball  $B_x$  centered at  $x$ ,
2. an index  $1 \leq i \leq n$ ,
3. a function  $h_x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  in  $C^{2,\gamma}(\mathbb{R}^{n-1})$  (see section C for the necessary definitions), with

$$\partial\Omega \cap B_x = \{z \in B_x : z_i = h_x(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)\} \quad (3.1)$$

and, either

$$\Omega \cap B_x = \{z \in B_x : z_i > h_x(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)\} \quad (3.2)$$

or

$$\Omega \cap B_x = \{z \in B_x : z_i < h_x(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)\} \quad (3.3)$$

Let  $f$  be a function in  $C^{0,\gamma}(\overline{\Omega} \times \mathbb{R})$  and  $\theta : \partial\Omega \rightarrow \mathbb{R}$  a continuous function. Consider the boundary value problem

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega \\ u(x) = \theta(x), & x \in \partial\Omega \end{cases} \quad (3.4)$$

Remember that  $\Delta u$  is called the *Laplacian of  $u$* , and it is defined as the real valued function

$$\Delta u(x) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x)$$

for each  $x \in \Omega$ . By a solution to this BVP we mean a function  $u \in C^0(\overline{\Omega})$  with  $u|_{\Omega} \in C^2(\Omega)$ , satisfying the two requirements in (3.4), that is,

$$\begin{cases} -\Delta u(x) = f(x, u(x)), & x \in \Omega \\ u(x) = \theta(x), & x \in \partial\Omega \end{cases} \quad (3.5)$$

The following is a key concept for the existence theory of problem (3.4).

**Definition 3** A function  $g \in C^0(\overline{\Omega})$  with  $g|_{\Omega} \in C^2(\Omega)$  is said to be a lower solution of (3.4), if

$$\begin{cases} -\Delta g(x) \leq f(x, g(x)), & x \in \Omega \\ g(x) \leq \theta(x), & x \in \partial\Omega \end{cases} \quad (3.6)$$

A function  $h \in C^0(\overline{\Omega})$  with  $h|_{\Omega} \in C^2(\Omega)$  is said to be an upper solution of (3.4), if

$$\begin{cases} -\Delta h(x) \geq f(x, h(x)), & x \in \Omega \\ h(x) \geq \theta(x), & x \in \partial\Omega \end{cases} \quad (3.7)$$

The following is the fundamental existence result for problem (3.4). A proof of this result, based on degree theoretic methods, can be found in [15].

**Theorem 4** If problem (3.4) admits a lower solution  $g$ , and an upper solution  $h$ , such that  $g(x) \leq h(x)$  for each  $x \in \overline{\Omega}$ , then there exists a solution  $u$  such that  $g(x) \leq u(x) \leq h(x)$  for each  $x \in \overline{\Omega}$ .

Let us consider the problem

$$\begin{cases} -\Delta u = \lambda F(x, u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \quad (3.8)$$

where  $F \in C^{0,\gamma}(\overline{\Omega} \times \overline{\mathbb{R}})$  with  $F(x, u) \geq 0$  for every  $(x, u) \in \overline{\Omega} \times \overline{\mathbb{R}}$  and  $\lambda$  is a fixed real number. We will denote this problem by  $P_{(F,\lambda)}$ . If we fix  $F$  but let  $\lambda$  vary over the reals we obtain a family of problems which we denote by  $P_{(F,\cdot)}$ . Notice that problem (3.8) is a particular case of problem (3.4), and the problem posed by the steady state model is in turn a particular case of problem (3.8). Due to the physical interpretation of  $u$  in the steady state model as the distribution of the absolute temperature on  $\overline{\Omega}$ , for our purpose it is important only to study the question of existence of nonnegative solutions to problem (3.8). The following definitions and results address this question.

**Lemma 5** *If  $\lambda \geq 0$ , then the function  $g \equiv 0$  is a lower solution for problem  $P_{(F,\lambda)}$ .*

**Proof.** Observe that  $g \equiv 0$  satisfies  $-\Delta g(x) = 0 \leq \lambda F(x, 0) = \lambda F(x, g(x))$  because  $\lambda \geq 0$  and  $F$  is a nonnegative function. Clearly  $g(x) \leq 0$  for each  $x \in \partial\Omega$ . ■

**Definition 6** *Fix  $F \in C^{0,\gamma}(\overline{\Omega \times \mathbb{R}})$  with  $F(x, u) \geq 0$  for every  $(x, u) \in \overline{\Omega \times \mathbb{R}}$ . The set  $\Sigma_F$  formed by those  $\lambda \in \mathbb{R}$  such that the corresponding problem  $P_{(F,\lambda)}$  has at least one nonnegative solution, is called the spectrum of problem  $P_{(F,\cdot)}$ .*

**Lemma 7** *If  $\lambda_1 \in \Sigma_F \cap (0, \infty)$ , then  $[0, \lambda_1] \subset \Sigma_F$ .*

**Proof.** Let  $\lambda \in [0, \lambda_1]$ . We need to see that problem  $P_{(F,\lambda)}$  admits a nonnegative solution. Since  $\lambda \geq 0$ , Lemma 5 says that  $g \equiv 0$  is a lower solution to  $P_{(F,\lambda)}$ . Let  $h$  be a nonnegative solution to problem  $P_{(F,\lambda_1)}$ . Such solution exists because  $\lambda_1 \in \Sigma_F$ . Let us see that  $h$  is an upper solution to problem  $P_{(F,\lambda)}$ . In fact,  $-\Delta h(x) = \lambda_1 F(x, h(x)) \geq \lambda F(x, h(x))$  for each  $x \in \Omega$ , and since  $h(x) = 0$  for each  $x \in \partial\Omega$ , we have in particular that  $h(x) \geq 0$  for each  $x \in \partial\Omega$ . We can now apply Theorem 4 obtaining the existence of a solution  $u$  to problem  $P_{(F,\lambda)}$  such that  $0 \leq u(x) \leq h(x)$  for each  $x \in \overline{\Omega}$ . In particular,  $u$  is a nonnegative solution for problem  $P_{(F,\lambda)}$  and therefore  $\lambda \in \Sigma_F$ . ■

The following lemma gives an upper bound for the set  $\Sigma_F$ . Its proof requires the following maximum principle.

**Theorem 8** *Let  $\Omega$  be a bounded connected open set in  $\mathbb{R}^n$  and  $u \in C^0(\overline{\Omega})$  with  $u|_{\Omega} \in C^2(\Omega)$ . The following hold:*

1. *If  $\Delta u(x) \geq 0$  for each  $x \in \Omega$  then  $u$  attains the value  $M := \max_{x \in \overline{\Omega}} u(x)$  at some point in  $\partial\Omega$ , and if  $u$  attains the value  $M$  at some point in  $\Omega$ , then  $u$  is a constant function.*
2. *If  $\Delta u(x) \leq 0$  for each  $x \in \Omega$  then  $u$  attains the value  $m := \min_{x \in \overline{\Omega}} u(x)$  at some point in  $\partial\Omega$ , and if  $u$  attains the value  $m$  at some point in  $\Omega$ , then  $u$  is a constant function.*

The proof of this theorem is presented in section 6.4 of [5].

**Lemma 9** *Assume that there exist functions  $f_0, r \in C^{0,\gamma}(\overline{\Omega})$  satisfying that  $f_0(x) > 0$  and  $r(x) > 0$  for every  $x \in \overline{\Omega}$ , such that*

$$F(x, u) \geq f_0(x) + r(x)u \quad (3.9)$$

for each  $(x, u) \in \overline{\Omega} \times [0, \infty)$ . Then the problem  $P_{(F, \lambda_1)}$  does not admit any nonnegative solution for  $\lambda_1 \geq \lambda_0(r)$ , where  $\lambda_0(r)$  is the smallest value of  $\lambda$  such that the problem

$$\begin{cases} -\Delta u = \lambda r(x)u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \quad (3.10)$$

admits a nontrivial (i.e. not identically zero) solution  $u \in C^0(\overline{\Omega})$  with  $u|_{\Omega} \in C^2(\Omega)$ .

**Remark 10** If we let  $v(x) = r(x)u(x)$  then problem (3.10) becomes

$$\begin{cases} Lv = \lambda v, & x \in \Omega \\ v = 0, & x \in \partial\Omega \end{cases} \quad (3.11)$$

where  $Lv = -\sum_{i=1}^n s(x)v_{x_i x_i} - \sum_{i=1}^n 2s_{x_i}(x)v_{x_i} - (\sum_{i=1}^n s_{x_i x_i}(x))v$  where  $s(x) = 1/r(x)$ . Since  $r$  is continuous and positive on the compact set  $\overline{\Omega}$ , there is an  $M > 0$  such that  $0 < r(x) < M$  for every  $x \in \overline{\Omega}$  and therefore  $1/M < s(x)$  for every  $x \in \overline{\Omega}$ . Since for each  $x \in \overline{\Omega}$  the diagonal matrix  $(s(x), \dots, s(x))$  is symmetric, positive definite, and has  $s(x)$  as its unique eigenvalue, we can conclude that  $L$  is an elliptic operator, because  $s(x) > 1/M > 0$  for each  $x \in \overline{\Omega}$ . For problems of the form (3.11) with  $L$  an elliptic operator, the set of  $\lambda \in \mathbb{R}$  for which there exists a nontrivial solution is called the spectrum of  $L$  and it is known to be a set of the form  $\lambda_0 < \lambda_1 < \lambda_2 < \dots$  and that there exists a positive  $\omega$  solution to problem 3.11 with  $\lambda = \lambda_0$ . This solution  $\omega$  is regular enough so that Green identities apply.

**Proof.** We begin the proof by observing that  $\lambda_0(r) > 0$ . For if  $\lambda_0(r) \leq 0$  then there would be a function  $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$  with  $u(x) \geq 0$  for each  $x \in \overline{\Omega}$ ,  $u(x_0) \neq 0$  for some  $x_0 \in \Omega$ , and satisfying  $-\Delta u(x) = \lambda_0(r)r(x)u(x)$  for each  $x \in \Omega$  and  $u(x) = 0$  for each  $x \in \partial\Omega$ . But since  $r(x) > 0$  for each  $x \in \Omega$ , we have that  $\lambda_0(r)r(x)u(x) \leq 0$  for each  $x \in \Omega$ . This implies that  $\Delta u(x) \geq 0$  for each  $x \in \Omega$ . Applying the maximum principle stated above (Theorem 8) to  $u$  we conclude that  $u$  attains its absolute maximum at some point  $x_0 \in \partial\Omega$ . But the fact that  $u$  vanishes on  $\partial\Omega$  allows us to infer that  $u(x) \leq 0$  for each  $x \in \overline{\Omega}$ . The latter conclusion contradicts the assumptions that  $u(x) \geq 0$  at each  $x \in \overline{\Omega}$  and that  $u$  does not vanish everywhere in  $\Omega$ .

Let  $h$  be a nonnegative solution to problem  $P_{(F, \lambda_1)}$  with  $\lambda_1 \geq \lambda_0(r)$ . Then

$$-\Delta h(x) = \lambda_1 F(x, h(x)) \geq \lambda_1 f_0(x) + \lambda_1 r(x)h(x) \quad (3.12)$$

for each  $x \in \Omega$ , and  $h(x) = 0$  for each  $x \in \partial\Omega$ . Also, the function  $g \equiv 0$  is clearly a solution to problem  $P_{(F, \lambda_1)}$ , and since  $f_0(x) > 0$  for each  $x \in \bar{\Omega}$  and  $\lambda_1 \geq \lambda_0(r) > 0$ , we have that

$$-\Delta g(x) = 0 \leq \lambda_1 f_0(x) = \lambda_1(f_0(x) + r(x)g(x)) \quad (3.13)$$

for every  $x \in \Omega$ .

Thus  $h$  is an upper solution and  $g$  is a lower solution of

$$\begin{cases} -\Delta u(x) &= \lambda_1(f_0(x) + r(x)u(x)), & x \in \Omega \\ u(x) &= 0, & x \in \partial\Omega \end{cases} \quad (3.14)$$

with  $h(x) \geq g(x)$  for each  $x \in \bar{\Omega}$  since  $h$  is nonnegative. By Theorem 4 there exists a solution  $u$  to problem (3.14) such that  $0 = g(x) \leq u(x) \leq h(x)$  for each  $x \in \bar{\Omega}$ . Notice that  $u$  is necessarily different from the zero function because this function is not a solution to problem (3.14). Indeed, if  $u \equiv 0$  were a solution then  $0 = -\Delta u(x) = \lambda_1(f_0(x) + r(x)u(x)) = \lambda_1 f_0(x)$  and therefore  $f_0(x) = 0$  for each  $x \in \bar{\Omega}$  because  $\lambda_1 \neq 0$ . This would contradict the positivity assumption about  $f_0$ . The fact that  $-\Delta u(x) > \lambda_1 r(x)u(x) \geq 0$ ,  $u$  does not vanish everywhere,  $u(x) = 0$  for each  $x \in \partial\Omega$  and  $\Omega$  is connected, allows us to conclude that  $u(x) > 0$  for each  $x \in \Omega$  as a consequence of part 2 of Theorem 8.

Let  $w$  be a nonnegative solution to problem (3.10) with  $\lambda = \lambda_0(r)$ , which does not vanish everywhere in  $\Omega$ . Now

$$\begin{aligned} 0 &= \int_{\Omega} (u\Delta w - w\Delta u) dV \\ &= \int_{\Omega} [u(-\lambda_0(r)r(x)w) - w(-\lambda_1(f_0(x) + r(x)u))] dV \\ &= \int_{\Omega} \{w[\lambda_1 f_0(x) + \lambda_1 r(x)u(x)] - u[\lambda_0(r)r(x)w(x)]\} dV \end{aligned} \quad (3.15)$$

where the first equality holds due to the fact that  $u$  and  $w$  vanish on  $\partial\Omega$ , and the second equality is a well known identity (see p.628 of [5]). This implies

$$(\lambda_0(r) - \lambda_1) \int_{\Omega} r(x)u(x)w(x)dV = \lambda_1 \int_{\Omega} w(x)f_0(x)dV. \quad (3.16)$$

The right hand side of the latter equation is strictly positive, since  $\lambda_1 > 0$ ,  $w$  is continuous, nonnegative and does not vanish everywhere in  $\Omega$ , and  $f_0(x) > 0$  for each  $x \in \Omega$ . On the other hand, the integral

$$\int_{\Omega} r(x)u(x)w(x)dV \quad (3.17)$$

is positive because  $r, u, w$  are continuous functions in  $\Omega$ , with  $r(x) > 0$  for each  $x \in \Omega$ , and  $u, w$  nonnegative and neither of them vanishing everywhere in  $\Omega$ . This combined with the positivity of the left hand side of equation (3.16) implies that  $\lambda_0(r) - \lambda_1 > 0$ , contradicting our assumption  $\lambda_1 \geq \lambda_0(r)$ . ■

A very important example to which the last Lemma applies is the so called *Gelfand problem* (see [9]):

$$\begin{cases} -\Delta u(x) &= \lambda e^{u(x)}, & x \in \Omega \\ u(x) &= 0, & x \in \partial\Omega \end{cases} \quad (3.18)$$

In this problem one can take  $f_0(x) = r(x) = 1$  for every  $x \in \bar{\Omega}$ , because  $e^u \geq 1 + u$  for any  $u \in [0, \infty)$ . Hence Lemma 9 allows us to conclude that if  $\lambda_1 \geq \lambda_0$ , where  $\lambda_0$  is the smallest value of  $\lambda$  for which problem

$$\begin{cases} -\Delta u(x) &= \lambda u(x), & x \in \Omega \\ u(x) &= 0, & x \in \partial\Omega \end{cases} \quad (3.19)$$

admits a nonconstant nonnegative solution, then problem (3.18) with  $\lambda = \lambda_1$  admits neither a nonconstant nor a nonnegative solution.

More generally, if in problem (3.8) the function  $F$  is such that  $F(x, 0) > 0$  for every  $x \in \bar{\Omega}$ ,  $F_u(x, u) > 0$  and  $F_{uu}(x, u) \geq 0$  for every  $(x, u) \in \bar{\Omega} \times [0, \infty)$ , then an immediate application of the one-dimensional Taylor theorem with residue shows that  $F(x, u) \geq F(x, 0) + F_u(x, 0)u$  for every  $(x, u) \in \bar{\Omega} \times [0, \infty)$ . Lemma 9 implies that if  $\lambda_1 \geq \lambda_0(F_u(\cdot, 0))$  then  $\lambda_1$  does not belong to  $\Sigma_F$ .

The next lemma is due to Bandle (see [1]) and uses symmetrization and isoperimetric inequalities.

**Lemma 11** *The (unique) solution  $w$  of*

$$\begin{cases} -\Delta w(x) &= 1, & x \in \Omega \\ w(x) &= 0, & x \in \partial\Omega \end{cases} \quad (3.20)$$

*satisfies*

$$w(x) \leq \frac{1}{2n} \left( \frac{V_n}{S_n} \right)^{\frac{2}{n}} \quad (3.21)$$

*for every  $x \in \bar{\Omega} \subset \mathbb{R}^n$ , where  $V_n$  and  $S_n$  denote the  $n$ -dimensional volumes of  $\Omega$  and the unit ball, respectively.*

As a consequence of this lemma, we have the following theorem.

**Theorem 12** *Assume that there exists a nonnegative nondecreasing function  $f_0 \in C^{0,\gamma}([0, \infty))$  such that  $F(x, u) \leq f_0(u)$  for every  $(x, u) \in \bar{\Omega} \times [0, \infty)$ . Suppose that the function  $\frac{m}{f_0(m)}$  defined for  $m \geq 0$  attains its absolute maximum at  $m_0$ . If*

$$\lambda_1 = \frac{2nm_0}{f_0(m_0)} \left( \frac{S_n}{V_n} \right)^{\frac{2}{n}} \quad (3.22)$$

then  $[0, \lambda_1] \subset \Sigma_F$  for problem (3.8).

**Proof.** Clearly the function  $g \equiv 0$  is a lower solution to problem  $P_{(F,\lambda)}$  for every  $\lambda \in \mathbb{R}$ . Fix  $\lambda \in [0, \lambda_1]$ , and define  $h(x) = \lambda f_0(m_0)w(x)$  for every  $x \in \bar{\Omega}$ .  $h$  is clearly a solution to problem

$$\begin{cases} -\Delta u(x) = \lambda f_0(m_0), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases} \quad (3.23)$$

Part 2) of Theorem 8 implies that  $w$  is a nonnegative function on  $\bar{\Omega}$ . As a consequence, the function  $h$  is also nonnegative on  $\bar{\Omega}$ . Furthermore,

$$h(x) = \lambda f_0(m_0)w(x) \leq \lambda_1 f_0(m_0) \frac{1}{2n} \left( \frac{V_n}{S_n} \right)^{\frac{2}{n}} = m_0 \quad (3.24)$$

for every  $x \in \bar{\Omega}$ , due to the upper bound for  $w$  given by Lemma 11. Now, we have

$$\lambda F(x, h(x)) \leq \lambda f_0(h(x)) \leq \lambda f_0(m_0) = -\Delta h(x) \quad (3.25)$$

for every  $x \in \Omega$ , because  $f_0$  is nondecreasing and inequality (3.24). But this fact combined with the nonnegativeness of  $h$  shows that  $h$  is an upper solution for  $P_{(F,\lambda)}$ . By Theorem 4 there exists a nonnegative solution of problem (3.8) and therefore  $\lambda \in \Sigma_F$ . We conclude that  $[0, \lambda_1] \subset \Sigma_F$ . ■

**Remark 13** *Notice that last proof never uses the fact that function  $\frac{m}{f_0(m)}$  defined for  $m \geq 0$  attains its absolute maximum at  $m_0$ . So the proof actually works for any choice of  $m_0 \geq 0$ . Now, since*

$$\lambda_1 = 2n \left( \frac{S_n}{V_n} \right)^{\frac{2}{n}} \frac{m_0}{f_0(m_0)}, \quad (3.26)$$

$\lambda_1$  is maximized when  $\frac{m_0}{f_0(m_0)}$  is maximized. So the choice of  $m_0$  in the hypothesis has the purpose of obtaining the largest possible interval inside  $\Sigma_F$  the argument is capable of producing.

Theorem 12 can be applied to the Gelfand problem when  $\Omega$  is the standard unit ball  $B(0, 1) \subset \mathbb{R}^n$ , with  $n = 1, 2, 3$ . In this case we can choose  $f_0(u) = e^u$  for  $u \geq 0$ . This function has a maximum value of  $e^{-1}$  and it is attained when  $m_0 = 1$ . Therefore  $\lambda_1 = 2ne^{-1}$ , for  $n = 1, 2, 3$ .

The next result is due to Kazdan and Warner [12].

**Theorem 14** *If  $F(x, u) > 0$  for  $(x, u) \in \bar{\Omega} \times [0, \infty)$ , then there is a  $\lambda_0 \in (0, \infty]$  such that for each  $\lambda \in (0, \lambda_0)$ , problem  $P_{(F, \lambda)}$  admits a solution which is strictly positive at each point of  $\Omega$ . If  $\lambda = 0$  then  $u \equiv 0$  is a solution of  $P_{(F, \lambda)}$ . If  $\lambda > \lambda_0$  problem  $P_{(F, \lambda)}$  admits no solution. If  $\lambda < 0$ , problem  $P_{(F, \lambda)}$  admits no nonnegative solution. In addition,*

1. *If  $\liminf_{s \rightarrow \infty} \frac{F(x, s)}{s} > 0$  uniformly in  $x \in \bar{\Omega}$ , then  $\lambda_0 < \infty$ .*
2. *If  $\lim_{s \rightarrow \infty} \frac{F(x, s)}{s} = 0$  uniformly in  $x \in \bar{\Omega}$ , then  $\lambda_0 = \infty$ .*

**Proof.** Let  $\lambda < 0$  and assume that  $u$  is a nonnegative solution of  $P_{(F, \lambda)}$ . Then  $-\Delta u(x) = \lambda F(x, u) \leq 0$  for  $x \in \Omega$  and  $u(x) = 0$  for  $x \in \partial\Omega$ . Applying part 1 of Theorem 8, we obtain that  $u(x) \leq 0$  for each  $x \in \Omega$ . But this contradicts one of our assumptions about  $u$ .

By Lemma 7 if  $\lambda_1 > 0$  is such that problem  $P_{(F, \lambda_1)}$  admits a nonnegative solution, then problem  $P_{(F, \lambda)}$  also admits a nonnegative solution, whenever  $\lambda \in [0, \lambda_1]$ . Let us define  $P$  as the set of real numbers  $\lambda$  such that problem  $P_{(F, \lambda)}$  admits a *positive* solution (i.e. a solution  $u$  with  $u(x) > 0$  for each  $x \in \Omega$ ). To see that  $P$  is not empty, we show that  $P_{(F, \lambda_1)}$  admits a positive solution for some  $\lambda_1 > 0$ . The function  $g \equiv 0$  is a lower solution of  $P_{(F, \lambda)}$  for each  $\lambda > 0$ . Let  $h$  be the solution to the problem

$$\begin{cases} -\Delta w(x) = 1, & x \in \Omega \\ w(x) = 0, & x \in \partial\Omega \end{cases} \quad (3.27)$$

(The last problem is a second order elliptic boundary value problem, and the existence and uniqueness of solutions for the latter is established in many references. See for example Chapter 6 of [5].) Notice that  $h$  cannot be constant (simply because such function is not a solution to (3.27)), and that  $\Delta h(x) \leq 0$  for each  $x \in \Omega$ . By part 2 of Theorem 8,  $h$  attains its absolute minimum at some point in  $\partial\Omega$ . But since  $h(x) = 0$  for each  $x \in \partial\Omega$  we have that the absolute minimum value of  $h$  is zero. Therefore  $h$  is nonnegative on  $\bar{\Omega}$ . Furthermore, part 2 of Theorem 8 implies that  $h(x) > 0$  for each  $x \in \Omega$  since  $h$  is not constant and  $\Omega$  is connected. The real valued function defined

on the compact set  $\overline{\Omega}$  sending  $x$  to  $F(x, h(x))$  is continuous. Therefore, there exists  $K > 0$  such that  $F(x, h(x)) \leq K$  for each  $x \in \overline{\Omega}$ . Now, if  $\lambda_1 > 0$  is such that  $1 \geq \lambda_1 K$ , then  $-\Delta h(x) = 1 \geq \lambda_1 F(x, h(x))$  for each  $x \in \overline{\Omega}$ . This combined with the fact that  $h(x) = 0 \geq 0$  for each  $x \in \partial\Omega$ , says that  $h$  is an upper solution of  $P_{(F, \lambda_1)}$ . Since it also holds that  $g(x) = 0 \leq h(x)$  for each  $x \in \overline{\Omega}$ , Theorem 4 implies that there is a solution  $u$  to problem  $P_{(F, \lambda_1)}$  satisfying  $0 \leq u(x) \leq h(x)$ . Let us see that  $u$  is positive. In fact, being a solution to  $P_{(F, \lambda_1)}$ ,  $u$  is not constant and  $\Delta u(x) = -\lambda_1 F(x, u(x)) \leq 0$  at each  $x \in \overline{\Omega}$ . Again, applying part 2 of Theorem 8 we conclude that  $u(x) > 0$  for each  $x \in \Omega$ . This shows that  $\lambda_1 \in P$  and therefore that  $P$ , actually  $P \cap (0, \infty)$ , is not empty. We define  $\lambda_0 = \sup P$ . We now proceed to prove parts 1 and 2.

1. For each  $s \in [1, \infty)$ , let us define  $\phi_s : \overline{\Omega} \rightarrow [0, \infty]$  by  $\phi_s(x) = F(x, s)/s$ . Let us see that the family  $\{\phi_s\}_{s \in [1, \infty)}$  is equicontinuous. By hypothesis, the function  $F : \overline{\Omega} \rightarrow (0, \infty)$  belongs to  $C^{0, \gamma}(\overline{\Omega})$ . In particular, there is a positive constant  $C > 0$ , such that

$$|F(x, s) - F(y, s)| \leq C \|x - y\|^\gamma \quad (3.28)$$

for every  $x, y \in \overline{\Omega}$  and  $s \geq 1$ . Since  $s \geq 1$ , inequality (3.28) implies

$$\left| \frac{F(x, s)}{s} - \frac{F(y, s)}{s} \right| \leq C \|x - y\|^\gamma \quad (3.29)$$

for every  $x, y \in \overline{\Omega}$  and  $s \geq 1$ . Now, for each  $\epsilon > 0$  take  $\delta(\epsilon)$  to be  $(\epsilon/C)^{1/\gamma}$ . It immediately follows that for every  $s \geq 1$ , the inequality  $\|x - y\| < \delta(\epsilon)$  implies the inequality  $|F(x, s)/s - F(y, s)/s| < \epsilon$ . But this is the equicontinuity of the family  $\{\phi_s\}_{s \in [1, \infty)}$ . So all the hypotheses necessary to apply Theorem 31 on appendices are fulfilled by  $\{\phi_s\}_{s \in [1, \infty)}$ . We conclude that there exist numbers  $\beta' > 0$  and  $s_1 \geq 1$ , such that  $F(x, s)/s = \phi_s(x) > \beta'$  or  $F(x, s) > \beta' s$ , for every  $s \geq s_1$  and  $x \in \overline{\Omega}$ . On the other hand, let  $m > 0$  be the absolute minimum value of the continuous function  $F$  in the compact set  $\overline{\Omega} \times [0, s_1]$ , and let  $\alpha = \min\{m, \beta' s_1\}$  and  $\beta = \alpha/s_1$ . It is easy to see that  $F(x, s) > \beta s + \alpha$  for each  $(x, s) \in \overline{\Omega} \times [0, \infty)$ .

Suppose now that  $u$  is a positive solution to  $P_{F, \lambda}$  and  $\psi$  a positive solution to the problem

2. If  $\lim_{s \rightarrow \infty} \frac{F(x, s)}{s} = 0$ ; since  $F(x, s) < s$  for  $s$  large enough and uniformly in  $x$ , an upper solution  $\bar{u}$  for any  $\lambda > 0$  can be constructed.

The function  $\underline{u} \equiv 0$  is always a lower solution. By Theorem 4, there is a solution  $\mu \geq 0$ , so  $\lambda = \sup P = \infty$ .

■

**Corollary 15** *Given any bounded domain  $\Omega \subset \mathbb{R}^n$ , there exists  $\delta_{FK} \in (0, \infty)$  such that*

1. *for  $0 < \delta < \delta_{FK}$ , BVP 2.52 has at least one positive solution, and*
2. *for  $\delta > \delta_{FK}$ , there is no solution.*  
*In addition, if  $\Omega$  is the unit ball in  $\mathbb{R}^n$ , then*

$$\frac{2n}{e} \leq \delta_{FK} \leq \frac{\mu}{e}. \quad (3.30)$$

*where  $\mu$  is the first eigenvalue of:  $-\Delta\psi = \mu\psi$  for  $x \in \partial\Omega$ .*

**Proof.** The existence of  $\delta_{FK}$  follows from Theorem 14. The lower bound on  $\delta_{FK}$  follows from Theorem 12. Since  $e^u \geq eu$  for all  $u \geq 0$ , the value of  $\beta$  in Theorem 14 can be chosen to be the number  $e$ ; the upper bound on  $\delta_{FK}$  follows. ■

The value  $\delta_{FK}$  is the critical value for the Frank-Kamenetski parameter  $\delta$ , which was used by the to differentiate between explosive and nonexplosive thermal events. For  $\delta > \delta_{FK}$ , the nonexistence of a solution for 2.52 was interpreted to mean that an explosion would occur.

## Chapter 4

# Radial Symmetry

Symmetrization techniques can be used to simplify a partial differential equation defined in a domain  $\Omega$  possessing certain symmetry properties. If  $\Omega$  is an open ball in  $\mathbb{R}^n$  centered at 0, then it could be seek radially symmetric solutions. Although this approach may not produce all solutions to a given problem associated with a given partial differential equation, Gidas, Ni, and Nirenberg proved in [10] that for a large class of problems, positive solutions are necessarily radially symmetric.

More precisely, for  $\Omega = \{x \in \mathbb{R}^n : |x| < R\} = B_R$ , let  $u \in C^2(\bar{\Omega})$  with  $u(x)$  defined in all reals, be a positive solution of

$$\begin{cases} -\Delta u = f(u), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases} \quad (4.1)$$

where  $f \in C^1(\mathbb{R})$  for  $f(x)$  defined in all reals; then  $u$  is radially symmetric and radially decreasing. That is, if  $r := |x|$ , then  $u = u(r)$  and  $u'(r) < 0$  for  $r \in (0, R)$ .

This implies that any positive solution of 4.1 is a solution of

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad 0 < r < R \quad (4.2)$$

$$u'(0) = 0, \quad u(R) = 0. \quad (4.3)$$

Thus, we need only determine the existence of positive solutions of the former two equations.

The assumption that  $u > 0$  is necessary. For example,  $u(x) = \sin(\pi x)$  is a solution to:  $-u'' = \pi^2 u$  for  $x \in (-1, 1)$  and  $u(\pm 1) = 0$ . We have  $u(x) > 0$  for  $x \in (0, 1)$ , but  $u(x) < 0$  for  $x \in (-1, 0)$ . The solution  $u(x)$  is not radially

symmetric. Even if  $u(x) \geq 0$  the full result may not be true. For example,  $u(x) = 1 - \cos(2\pi x)$  is a solution to:  $-u'' = 4\pi^2(u - 1)$  for  $x \in (-1, 1)$  and  $u(\pm 1) = 0$ . We have  $u(x) \geq 0$  (where  $u(0) = 0$ ) and  $u(-x) = u(x)$ , but  $u(x)$  is not radially decreasing. Note that the condition  $f(u) \geq 0$  for all  $u$  implies that any nontrivial solution is positive (by the strong maximum principle, referenced in [5], [14] and ).

Although the result is stated for  $f \in C^1$ , this hypothesis can be weakened. The result also holds for any function  $f = f_1 + f_2$  where  $f_1 \in C^1$  and  $f_2$  is monotone increasing. In particular, the result holds if  $f$  is locally Lipschitz continuous.

The proof utilizes maximum principles and the method of moving parallel planes. We first prove a maximum principle which is more delicate to prove than the standard ones. It is a generalization of the Hopf lemma.

**Lemma 16** *Let  $\Omega^*$  be a bounded domain whose boundary  $\partial\Omega^*$  is of class  $C^2$ . Let  $T$  be a hyperplane containing the normal to  $\partial\Omega^*$  at some point  $q$ . Let  $\Omega$  be that portion of  $\Omega^*$  which lies on one side of  $T$ . Let  $w \in C^2(\overline{\Omega})$  satisfy  $w(x) \geq 0$  and  $\Delta w(x) \leq 0$  for each  $x \in \Omega$ , and  $w(q) = 0$ . If  $s$  is any direction vector at  $q$  entering  $\Omega$  nontangentially, then*

$$\frac{\partial w}{\partial s}(q) > 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2}(q) > 0 \quad (4.4)$$

*unless  $w \equiv 0$  on each connected component of  $\Omega$  whose boundary contains  $q$ .*

**Proof.** Without loss of generality, we can orient  $\Omega^*$  so that the plane  $T$  has normal vector  $\gamma = (1, 0, \dots, 0)$ . Let  $\Omega$  be on the side of  $T$  which  $\gamma$  points to. Let  $K_1$  be an open ball internally tangent to  $\Omega^*$  at  $q$  with radius  $r_1$ . Without loss of generality, translate  $\Omega^*$  so that the origin  $0 \in \mathbb{R}^n$  becomes the center of  $K_1$ . Let  $K_2$  be the open ball of radius  $\frac{1}{2}r_1$  centered at  $q$ . Define  $K = K_1 \cap K_2 \cap \Omega$ . Figure 4.1 illustrates these sets.

Define  $z(x) = x_1(e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2})$  where  $(r(x))^2 = x_1^2 + \dots + x_n^2$  is the square of the distance from  $x$  to the origin, and  $\alpha > 0$  is a constant which will be adjusted soon. Let us see that

$$\Delta z = 2\alpha x_1 e^{-\alpha(r(x))^2} [2\alpha(r(x))^2 - (n+2)], \quad x \in \mathbb{R}^n. \quad (4.5)$$

In fact, we have

$$\frac{\partial z(x)}{\partial x_1} = (e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2}) - 2\alpha x_1^2 (e^{-\alpha(r(x))^2})$$

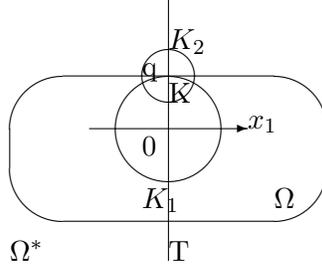


Figure 4.1: Radial Symmetry

and

$$\begin{aligned} \frac{\partial^2 z(x)}{\partial x_1^2} &= e^{-\alpha(r(x))^2}(-2\alpha x_1) - e^{-\alpha(r(x))^2}(-4\alpha x_1) + e^{-\alpha(r(x))^2}(4\alpha^2 x_1^2) \\ &= 2\alpha x_1 e^{-\alpha(r(x))^2}(2\alpha x_1^2 - 3) \end{aligned}$$

For  $i = 2, \dots, n$ ,

$$\frac{\partial z(x)}{\partial x_i} = -2\alpha x_1 x_i e^{-\alpha(r(x))^2}$$

and

$$\frac{\partial^2 z(x)}{\partial x_i^2} = 2\alpha x_1 e^{-\alpha(r(x))^2}(2\alpha x_i^2 - 1).$$

Consequently,

$$\begin{aligned} \Delta z &= 2\alpha x_1 e^{-\alpha(r(x))^2}((2\alpha x_1^2 - 3) + (2\alpha x_2^2 - 1) + \dots + (2\alpha x_n^2 - 1)) \\ &= 2\alpha x_1 e^{-\alpha(r(x))^2}[2\alpha(r(x))^2 - (n+2)]. \end{aligned}$$

The function  $z$  satisfies

$$z(x) > 0, x \in K \quad \text{and} \quad z(x) = 0, x \in T \cup \partial K_1 \quad (4.6)$$

The first property is justified by the fact that for all  $x \in K$ ,  $x_1 > 0$  and  $r(x) < r_1$ . The second one follows from the facts that  $x_1 = 0$  for each  $x \in T$ , and  $r(x) = r_1$  for each  $x \in \partial K_1$ . Now, if we choose  $\alpha = 2(n+2)/r_1^2$ , we claim that  $\Delta z(x) > 0$  for each  $x \in K$ . In fact, since  $2\alpha x_1 e^{-\alpha(r(x))^2} > 0$  for all  $x \in K$ , it suffices to check that  $2\alpha(r(x))^2 - (n+2) > 0$  for the chosen  $\alpha$ . Since

$$\frac{r_1^2}{4} < (r(x))^2$$

we have

$$\frac{1}{(r(x))^2} < \frac{4}{r_1^2}$$

Multiplying by  $(n+2)/2$  we arrive at

$$\frac{(n+2)}{2(r(x))^2} < \frac{2(n+2)}{r_1^2} = \alpha$$

Therefore

$$2\alpha(r(x))^2 - (n+2) > 0$$

Let us assume that  $w \neq 0$  on some connected component  $\Omega_1$  of  $\Omega$ , satisfying  $q \in \partial\Omega_1$ . Let us see that  $w(x) > 0$  for each  $x$  in  $\Omega_1$ . By hypothesis  $w(x) \geq 0$  for each  $x$  in the connected set  $\overline{\Omega}_1$  and  $\Delta w(x) \geq 0$  for each  $x \in \Omega_1$ . The strong maximum principle tells us that the function  $w$  restricted to  $\overline{\Omega}_1$  reaches its absolute minimum *only* on the boundary of  $\Omega_1$ . Now, since  $w(q) = 0$  and  $q \in \partial\Omega_1$ , the absolute minimum of  $w$  on  $\overline{\Omega}_1$  is 0. This implies that  $w(x)$  is never zero for  $x \in \Omega_1$ , and as a consequence that  $w(x) > 0$  for each  $x \in \Omega_1$ .

Let us denote the set  $\partial K \cap \partial K_2$  by  $A$ . We observe that  $A$  is a compact set. We claim that there exists  $\epsilon > 0$  such that  $w(x) > \epsilon x_1$  for every  $x \in A$ . Suppose that this is not the case, that is, that there exists a sequence  $\{p_n\}_{n=1}^\infty$  of points in  $A$  such that

$$w(p_n)/x_1(p_n) < 1/n \tag{4.7}$$

Notice that it is not possible that all the terms of  $\{p_n\}_{n=1}^\infty$  satisfy  $x_1(p_n) \geq C$  for some fixed constant  $C > 0$ , because the function  $w/x_1$ , being continuous and strictly positive on the compact set  $S := A \cap \{p \in \mathbb{R}^n : x_1(p) \geq C\}$ , has a positive absolute minimum in  $S$ , in contradiction with property (4.7). Let us extract a subsequence  $\{p_{n_k}\}_{k=1}^\infty$  such that  $x_1(p_{n_k})$  goes to zero as  $k$  goes to infinity. Since  $A$  is compact we can extract a subsequence  $\{p_{n_{k_l}}\}_{l=1}^\infty$

converging to a point  $p \in A$ . It is clear that  $x_1(p)$  must be equal to zero. Let us see that  $w(p) = 0$ . This immediately follows from the fact that

$$w(p_{n_{k_l}}) < (1/n_{k_l})x_1(p_{n_{k_l}}) \quad (4.8)$$

for each  $l$ , and the fact that the right hand side approaches zero as  $l$  goes to  $\infty$ . Since  $\Delta w(x) \leq 0$  for each  $x \in \Omega$ , and  $w(x) > 0$  for each  $x \in \Omega$  and therefore  $w(p) < w(x)$  for each  $x \in \Omega$  with  $p \in \partial\Omega$ , Hopf's lemma applies, allowing us to conclude that  $\frac{\partial w}{\partial \nu}(p) < 0$ . But since  $\frac{\partial w}{\partial \nu}(p) = -\frac{\partial w}{\partial x_1}(p)$ , one obtains  $\frac{\partial w}{\partial x_1}(p) > 0$ . Now, the continuity of  $\frac{\partial w}{\partial x_1}$  on  $\bar{\Omega}$  combined with the facts that  $x_1(p) = 0$  and the distance between  $p$  and  $q$  is  $r_1/2 > 0$  guarantees the existence of an open ball  $B(p, r)$  such that  $U := B(p, r) \cap \{x \in \mathbb{R}^n : x_1 > 0\} \subset \Omega$  and  $\frac{\partial w}{\partial x_1}(x) > 0$  for each  $x \in \bar{U}$ . The compactness of  $\bar{U}$  and the continuity of  $\frac{\partial w}{\partial x_1}$  on  $\bar{\Omega}$  implies the existence of a number  $K > 0$  such that  $\frac{\partial w}{\partial x_1}(x) \geq K$  for each  $x \in \bar{U}$ . For each  $x = (x_1, \dots, x_n) \in U$  let  $x'$  denote the point  $(0, x_2, \dots, x_n)$ . We have

$$\begin{aligned} w(x) &= w(x') + \int_0^{x_1} \frac{\partial w}{\partial x_1}(t, x_2, \dots, x_n) dt \\ &\geq \int_0^{x_1} \frac{\partial w}{\partial x_1}(t, x_2, \dots, x_n) dt \\ &\geq Kx_1 \end{aligned} \quad (4.9)$$

for each  $x \in \bar{U}$ . On the other hand, since  $p_{n_{k_l}}$  converges to  $p$  as  $l$  goes to infinity, there exists an  $L$  such that if  $l \geq L$  then  $p_{n_{k_l}} \in \bar{U}$ , and (4.9) implies that  $w(p_{n_{k_l}}) \geq Kx_1(p_{n_{k_l}})$ . This conclusion contradicts (4.8), allowing us to infer that there exists  $\epsilon > 0$  such that  $w(x) > \epsilon x_1$  for each  $x \in A$ . By hypothesis,  $w(x) \geq 0$  for each  $x \in \bar{\Omega}$  and therefore  $w(x) \geq 0$  for each  $x \in (\partial K \cap \partial K_1) \cup (\partial K \cap T)$ . Let us see that  $z(x) \leq x_1$  for every  $x \in A$ . This amounts to prove that  $e^{-\alpha r^2(x)} - e^{-\alpha r_1^2} = e^{-2(n+2)(r(x)/r_1)^2} - e^{-2(n+2)} < 1$  for each  $x \in A$  and each integer  $n \geq 1$ . The latter inequality follows by writing  $e^{-2(n+2)(r(x)/r_1)^2} - e^{-2(n+2)} = e^{-2(n+2)(r(x)/r_1)^2} (1 - e^{-2(n+2)(1-(r(x)/r_1)^2)})$  and observing that  $\frac{1}{2} \leq \frac{r(x)}{r_1} \leq 1$  for each  $x \in A$ .

Define  $\phi(x) = \omega(x) - \epsilon z(x)$  for each  $x \in \bar{K}$ . Then  $\phi$  satisfies the following three conditions:

$$\phi(x) \geq 0 \text{ for } x \in \partial K, \quad (4.10)$$

$$\phi(q) = 0 \quad (4.11)$$

$$\Delta \phi(x) < 0 \text{ for } x \in K \quad (4.12)$$

It is clear that  $\phi(x) \geq 0$  for each  $x \in T \cup \partial K_1$  since  $z(x) = 0$  for each  $x$  in this set. On the other hand, we know that  $w(x) \geq \epsilon x_1$  for each  $x \in A$  and  $x_1 \geq z$

in general for all  $x \in \overline{K}$ . On the definition of  $z(x) = x_1(e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2})$  the last sentence means that  $(e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2}) < 1$  and with our selection of  $\alpha$

$$e^{-2(n+2)(\frac{r}{r_1})^2} - e^{-2(n+2)} < 1$$

but

$$\frac{r}{r_1} \geq \frac{r_1}{2r_1} = \frac{1}{2}$$

and

$$\left(\frac{r}{r_1}\right)^2 \geq \frac{1}{4}$$

In general, it is valid that for  $0 < a < 1$

$$0 < a^{1/4} - a < 1$$

but

$$(a^{1/4} - a) = (a(a^{-3/4} - 1)) = a\left(\frac{1 - a^{3/4}}{a^{3/4}}\right) = a^{1/4}(1 - a^{3/4})$$

so

$$1 > a^{1/4} - a \geq a^t - a \text{ for } t \geq \frac{1}{4}$$

As final result,  $w(x) \geq \epsilon x_1 \geq \epsilon z(x)$ ;  $\therefore w(x) \geq \epsilon z(x)$  for  $x \in A$ .

We claim that  $\phi(q) = 0$ , because  $w(q) = 0$  by definition and  $z(q) = 0$  because of the form of  $z(x)$ .

Finally  $\Delta\phi < 0$  because  $\Delta w \leq 0$ ,  $\Delta z > 0$  and  $\epsilon$  is positive, so  $\Delta\phi = \Delta w - \epsilon\Delta z < 0$  on  $K$ . By the strong maximum principle,  $\phi > 0$  on  $K$ . At  $q$  we have either  $\phi_s > 0$  or  $\phi_{ss} \geq 0$ ; because if we define

$$f(t) = \phi(\vec{q} + t\vec{s})$$

the last requirement is the same as

$$f'(0) > 0 \text{ or } f''(0) \geq 0.$$

If that were not the case for  $f : [0, c)$  on  $\mathbb{R}$  then

$$f'(0) \leq 0 \text{ and } f''(0) < 0.$$

but that can not be because if  $f''(0) < 0$ , there exists  $c > 0$  such that  $f''(t)$  for  $0 \leq t < c$ , so  $f'$  is strictly decreasing on  $[0, c)$ . In other words, as  $f'(0) \leq 0$  then  $f'(t) < 0$  if  $t \in [0, c)$  and  $f$  is decreasing on  $[0, c)$ ; but  $f(0) = 0$  then  $f(t) < 0$  for  $t \in (0, c)$ . That is a contradiction because  $f(t) < 0$  implies  $\phi(\vec{q} + t\vec{s}) < 0$  and we had assumed  $\phi(x) > 0$  for all  $x \in K$ .

Let us see now that  $z_s(q) = 0$  and  $z_{ss}(q) > 0$ , so either  $w_s(q) > 0$  or  $w_{ss}(q) > 0$ . By definition  $z(x) = x_1(e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2})$  and also can be established that  $z_s(\vec{q} + t\vec{s}) = \nabla z(\vec{q} + t\vec{s}) \cdot \vec{s}$ ; on the other hand we have

$$\frac{\partial z(x)}{\partial x_1} = (e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2}) - 2\alpha x_1^2(e^{-\alpha(r(x))^2})$$

and specifically on  $q$

$$\frac{\partial z(q)}{\partial x_1} = (e^{-\alpha(r(q))^2} - e^{-\alpha r_1^2}) - 2\alpha x_1(q)^2(e^{-\alpha(r(q))^2}) = 0$$

as a consequence of  $r(q) = r_1$  on one side and  $x_1(q) = 0$  by the other. Also we established before, for  $i = 2, \dots, n$ ,

$$\frac{\partial z(x)}{\partial x_i} = -2\alpha x_1 x_i e^{-\alpha(r(x))^2}$$

that evaluated at  $q$  results in

$$\frac{\partial z(q)}{\partial x_i} = -2\alpha x_1(q) x_i(q) e^{-\alpha(r(q))^2} = 0$$

because of  $x_1(q) = 0$ .

In  $t = 0$  we have  $z_s(\vec{q}) = \nabla z(\vec{q}) \cdot \vec{s}$ . Which also means

$$z_s(\vec{q}) = \left( \frac{\partial z(q)}{\partial x_1}, \dots, \frac{\partial z(q)}{\partial x_n} \right) \cdot \vec{s} = 0$$

If we claim now  $f(t) = z_s(\vec{q} + t\vec{s})$  because of what has been seen above

$$\frac{d}{dt} f(t) = \frac{d}{dt} (z_s(\vec{q} + t\vec{s})) = \frac{d}{dt} (\nabla z(\vec{q} + t\vec{s}) \cdot \vec{s})$$

Evaluating  $\nabla z$  on  $\vec{q} + t\vec{s}$  we have for

$$\frac{\partial z(x)}{\partial x_1} = (e^{-\alpha(r(x))^2} - e^{-\alpha r_1^2}) - 2\alpha x_1^2(e^{-\alpha(r(x))^2})$$

so that

$$\frac{\partial z(\vec{q} + t\vec{s})}{\partial x_1} = (1 - 2\alpha(t s_1)^2) e^{-\alpha(r(\vec{q} + t\vec{s}))^2} - e^{-\alpha(r_1)^2}$$

and for  $i = 2, \dots, n$ ,

$$\frac{\partial z(x)}{\partial x_i} = -2\alpha x_1 x_i e^{-\alpha(\vec{q} + t\vec{s})^2}$$

so that

$$\frac{\partial z(\vec{q} + t\vec{s})}{\partial x_i} = -2\alpha(ts_1)(q_i + ts_i)e^{-\alpha(\vec{q}+t\vec{s})^2}$$

In that way  $\frac{d}{dt}(\nabla z(\vec{q} + t\vec{s}) \cdot \vec{s})$  evaluated at zero will be

$$\begin{aligned} \frac{d}{dt} & ((1 - 2\alpha(ts_1)^2)e^{-\alpha(r(\vec{q}+t\vec{s}))^2} - e^{-\alpha(r_1)^2}, \\ & - 2\alpha(ts_1)(q_2 + ts_2)e^{-\alpha(r(\vec{q}+t\vec{s}))^2}, \dots, \\ & - 2\alpha(ts_1)(q_n + ts_n)e^{-\alpha(r(\vec{q}+t\vec{s}))^2}) \end{aligned}$$

that evaluated at  $q$ , where  $t = 0$  results in

$$\begin{aligned} & (-2\alpha e^{-\alpha(r_1)^2}(\vec{q} \cdot \vec{s}), -2\alpha s_1 q_2 e^{-\alpha(r_1)^2}, \dots, -2\alpha s_1 q_n e^{-\alpha(r_1)^2}) \cdot \vec{s} > 0 \\ & (-2\alpha(\vec{q} \cdot \vec{s}), -2\alpha s_1 q_2, \dots, -2\alpha s_1 q_n) \cdot (s_1, \dots, s_n) > 0 \end{aligned}$$

but we know that  $s_1 > 0$  and that  $q \cdot \vec{s} < 0$  (vector entering), so that

$$((\vec{q} \cdot \vec{s}), s_1 q_2, \dots, s_1 q_n) \cdot (s_1, \dots, s_n) < 0$$

■

## 4.1 The Method of Moving Parallel Planes

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ . Let  $\lambda \in \mathbb{R}$  and let  $\gamma \in \mathbb{R}^n$  be a unit vector. Let us define  $T_\lambda = \{x \in \mathbb{R}^n : \gamma \cdot x = \lambda\}$  to be the hyperplane with normal  $\gamma$  and whose distance from the origin  $0$  is  $|\lambda|$ . There is a  $\lambda_0$  sufficiently large such that  $T_{\lambda_0} \cap \bar{\Omega} \neq \emptyset$  and  $T_\lambda \cap \bar{\Omega} = \emptyset$  for  $\lambda > \lambda_0$ . For any  $x \in \mathbb{R}^n$ , let  $x^\lambda$  be its reflection through  $T_\lambda$ .

Let us define  $\Sigma(\lambda) = \Omega \cap \{x : \gamma \cdot x > \lambda\}$ ; then  $\Sigma(\lambda) = \emptyset$  for  $\lambda \geq \lambda_0$  and  $\Sigma(\lambda) \neq \emptyset$  for  $\lambda < \lambda_0$ . The set  $\Sigma(\lambda)$  is called an *open cap*. Let us define  $\Sigma'(\lambda)$  to be the reflection of  $\Sigma(\lambda)$  through the plane  $T_\lambda$ . Like shown in Figure 4.2.

For  $\lambda < \lambda_0$  with  $|\lambda - \lambda_0|$  sufficiently small, it can be seen that  $\Sigma'(\lambda) \subseteq \Omega$ . Decreasing  $\lambda$  further, we have  $\Sigma'(\lambda) \subseteq \Omega$  until either

1.  $\Sigma'(\lambda)$  becomes internally tangent to  $\partial\Omega$  at some  $p \notin T_\lambda$ , or
2.  $T_\lambda$  is orthogonal to  $\partial\Omega$  at some  $q \in T_\lambda \cap \partial\Omega$ .

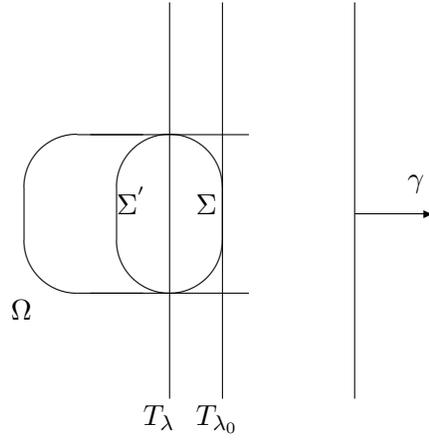


Figure 4.2: Moving Parallel Planes Method

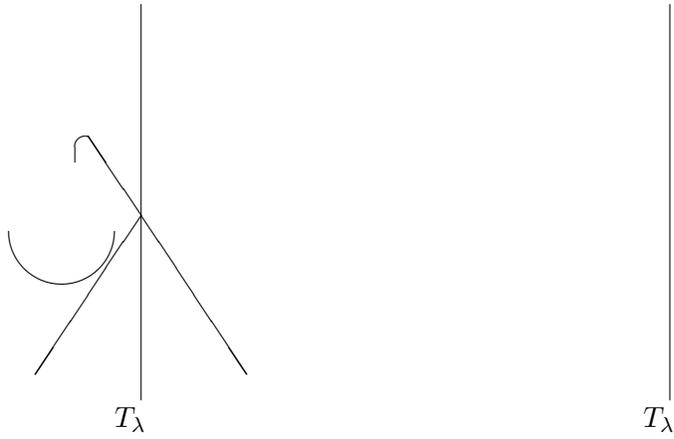


Figure 4.3: Examples of conditions 1. and 2.

That conditions are shown in Figure 4.3.

Define

$$\lambda_1 = \sup\{\lambda < \lambda_0 : \text{condition 1. or 2. occurs}\}.$$

The cap  $\Sigma(\lambda_1)$  is called the *maximal cap* associated with  $\gamma$ . Note that  $\Sigma'(\lambda) \subseteq \Omega$ , if  $\lambda \in [\lambda_1, \lambda_0)$ .

If  $\lambda$  is decreased below  $\lambda_1$ , it may be that  $\Sigma'(\lambda) \subseteq \Omega$ . Let us define

$$\lambda_2 = \inf\{\lambda < \lambda_0 : \Sigma'(\bar{\lambda}) \subseteq \Omega \text{ for } \bar{\lambda} \in (\lambda, \lambda_0)\}.$$

The cap  $\Sigma(\lambda_2)$  is called the *optimal cap* associated with  $\gamma$ . Observe that at  $\lambda_2$  either 1. or 2. occurs and  $\Sigma'(\lambda_2) \subseteq \Omega$ .

Without loss of generality it can be assumed that  $\gamma = (1, 0, \dots, 0) \in \mathbb{R}^n$  and  $\lambda_0 = \max\{x_1 : x \in \bar{\Omega}\}$  where  $x = (x_1, \dots, x_n)$ . Let  $\lambda_1$  and  $\lambda_2$  be defined as above. Define  $\Sigma_1$  to be the maximal cap associated with  $\gamma$  and denote its reflection through  $T_{\lambda_1}$  by  $\Sigma_1'$ . Let us define  $\Sigma_2$  to be the optimal cap associated with  $\gamma$  and denote its reflection through  $T_{\lambda_2}$  by  $\Sigma_2'$ .

For  $x_0 \in \partial\Omega$  and  $\varepsilon > 0$ , define a *neighborhood of  $x_0$*  in  $\Omega$  by  $\Omega_\varepsilon = \Omega \cap B_\varepsilon(x_0)$  where  $B_\varepsilon(x_0)$  is the ball of radius  $\varepsilon$  centered at  $x_0$ . Let us define  $S_\varepsilon = \partial\Omega \cap B_\varepsilon(x_0)$ . Let  $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$  be the unit outward normal to  $\partial\Omega$  at  $x$ .

**Lemma 17** *Let  $x_0 \in \partial\Omega$  be such that  $\nu(x_0) \cdot \gamma > 0$ . Choose  $\varepsilon > 0$  sufficiently small so that  $\nu(x) \cdot \gamma > 0$  for all  $x \in S_\varepsilon$ . If  $u \in C^2(\bar{\Omega}_\varepsilon)$  satisfies  $u_{x_1}(x_0) = 0$ ,  $u(x) = 0$  for  $x \in S_\varepsilon$ , and  $u(x) > 0$  for  $x \in \Omega_\varepsilon$ , then*

$$\nabla u(x_0) = 0$$

and

$$D^2u(x_0) = [\Delta u(x_0)]\nu(x_0)\nu^t(x_0)$$

where  $D^2u = [u_{x_i x_j}]$  is the  $n \times n$  matrix of second derivatives of  $u$ .

**Proof.** On  $S_\varepsilon$  we know that  $u(x) \equiv 0$  and so  $\nabla u(x)$  is normal to  $S_\varepsilon$  at each  $x$ . Since  $S_\varepsilon$  is a smooth  $(n-1)$ -dimensional manifold as described in [11], the tangent space  $T(x)$  to  $x \in S_\varepsilon$  is  $(n-1)$ -dimensional, meaning that

$$T(x) = \text{span}\langle w^1(x), \dots, w^{n-1}(x) \rangle,$$

where the  $w^k(x)$  form an orthonormal set for each  $x$ . As consequence,  $\nabla u(x) \cdot w^k(x) = 0$  for  $k = 1, \dots, n-1$ . Since by hypothesis  $\nu(x) \cdot \gamma > 0$  on  $S_\varepsilon$  it must be that  $\gamma \notin T(x)$  and so  $\{w^1(x), \dots, w^{n-1}(x), \gamma\}$  is a basis for  $\mathbb{R}^n$  for each

$x$ . The basis coefficients for  $\nabla u(x_0)$  are given by  $\nabla u(x_0) \cdot w^k(x_0) = 0$  for  $k = 1, \dots, n-1$ , and  $\nabla u(x_0) \cdot \gamma = u_{x_1}(x_0) = 0$ . Consequently,  $\nabla u(x_0) = 0$ . Let  $x(s)$  be any smooth curve on  $S_\varepsilon$  such that  $x(0) = x_0$ . Since  $u \equiv 0$  on  $S_\varepsilon$ , we have

$$y(s)^T \nabla u(x(s)) \equiv 0$$

for any smooth function  $y(s) \in T(x(s))$  and considering that the gradient is orthogonal to tangent plane. Differentiating with respect to  $s$  gives us

$$y(s)^T D^2 u(x(s)) x'(s) + y'(s)^T \nabla u(x(s)) \equiv 0.$$

It can be chosen  $n-1$  curves  $x(s)$  so that at  $s = 0$ ,

$$w^i(x_0)^T D^2 u(x_0) w^j(x_0) = 0, \quad (4.13)$$

with  $i, j = 1, \dots, n-1$ .

The hypotheses on  $u$  guarantee that  $\frac{\partial u(x)}{\partial \nu} \leq 0$  for  $x \in S_\varepsilon$ . Moreover, since  $\nabla u(x)$  and  $\nu(x)$  are parallel, we have

$$\nu(x(s))^T \nabla u(x(s)) = -|\nabla u(x(s))| =: -p(s)$$

for any smooth curve  $x(s)$  on  $S_\varepsilon$  with  $x(0) = x_0$ . The function  $p(s)$  is nonnegative and differentiable. Thus, at a point where  $p = 0$ , we must have  $p' = 0$ , a local minimum. In particular,  $p'(0) = 0$  since we had proved  $\nabla u(x_0) = 0$ . Differentiating with respect to  $s$  gives us

$$\left[ \frac{d}{ds} \nu(x(s)) \right]^T \nabla u(x(s)) + \nu(x(s))^T D^2 u(x(s)) x'(s) = -p'(s).$$

At  $s = 0$  we have  $\nu(x_0)^T D^2 u(x_0) x'(0) = 0$ . The curves  $x(s)$  can be chosen to obtain

$$\nu(x_0)^T D^2 u(x_0) w^j(x_0) = 0, \quad (4.14)$$

for  $j = 1, \dots, n-1$ . The set  $\{w^1(x_0), \dots, w^{n-1}(x_0), \nu(x_0)\}$  is orthonormal, so the block matrix

$$Q(x_0) = [w^1(x_0) | \dots | w^{n-1}(x_0) | \nu(x_0)]$$

is orthogonal,  $Q(x_0)e^k = w^k(x_0)$  for  $k = 1, \dots, n-1$  and  $Q(x_0)e^n = \nu(x_0)$ , where the  $e^k$  are the standard Euclidean basis vector in  $\mathbb{R}^n$ .

Combining equations (4.13) and (4.14), we obtain

$$m_{ij} = (e^i)^T \left( Q(x_0)^T D^2 u(x_0) Q(x_0) \right) e^j = 0$$

for  $i = 1, \dots, n$  and  $j = 1, \dots, n - 1$ . Since  $D^2u(x_0)$  is symmetric because of Clariaut theorem; from linear algebra it can be see that

$$Q^T(x_0)D^2u(x_0)Q(x_0) = \text{diag}\{0, \dots, 0, m_{nn}(x_0)\}.$$

Similar matrices have the same trace, so

$$\text{trace}(Q(x_0)^T D^2u(x_0) Q(x_0)) = \text{trace}(D^2u(x_0)).$$

That is,  $m_{nn}(x_0) = \text{trace}(D^2u(x_0)) = \Delta u(x_0)$ . Finally giving

$$D^2u(x_0) = Q(x_0)\text{diag}\{0, \dots, 0, \Delta u(x_0)\}Q^T(x_0) = \nu(x_0)\Delta u(x_0)\nu^T(x_0)$$

completing that way the proof. ■

**Lemma 18** *Let  $x_0 \in \partial\Omega$  be such that  $\nu(x_0) \cdot \gamma > 0$ . Choose  $\varepsilon > 0$  sufficiently small so that  $\nu(x) \cdot \gamma > 0$  for all  $x \in S_\varepsilon$ . Assume that  $u \in C^2(\bar{\Omega}_\varepsilon)$  satisfies*

1.  $u(x) > 0$  for  $x \in \Omega$ ,
2.  $\Delta u + f(u) = 0$  for  $x \in \Omega_\varepsilon$ , and
3.  $u(x) = 0$  for  $x \in S_\varepsilon$ ; then there is a  $\delta \in (0, \varepsilon)$  such that  $u_{x_1} < 0$  on  $\Omega_\delta$ .

**Proof.** Since  $u > 0$  on  $\Omega$  and  $u = 0$  on  $S_\varepsilon$ , it is necessary that  $\nabla u \cdot w \leq 0$  on  $S_\varepsilon$  for any vector  $w$  such that  $\nu \cdot \omega > 0$ . In particular, since  $\nu(x) \cdot \gamma > 0$ , we must have  $u_{x_1}(x) = \nabla u(x) \cdot \gamma \leq 0$  on  $S_\varepsilon$ .

If the conclusion is false, then there is a sequence  $\{x^j\}_{j=1}^\infty \subseteq \Omega_\varepsilon$  such that  $x^j \rightarrow x_0$  as  $j \rightarrow \infty$  and  $u_{x_1}(x^j) \geq 0$ . For  $j$  large, the interval  $I_j \subset \mathbb{R}^n$  in the  $x_1$ -direction from  $x^j$  to  $\partial\Omega$  intersects  $S_\varepsilon$  at  $a^j$  with  $u_{x_1}(a^j) \leq 0$ . Thus, there exists a sequence  $\{\bar{x}^j\}_{j=1}^\infty \subseteq \Omega_\varepsilon$  such that  $\bar{x}^j \rightarrow x_0$  as  $j \rightarrow \infty$  and  $u_{x_1}(\bar{x}^j) = 0$ .

By continuity we get

$$u_{x_1}(x_0) = \lim_{j \rightarrow \infty} u_{x_1}(\bar{x}^j) = 0. \quad (4.15)$$

By the Mean Value Theorem, there is a sequence  $\{\hat{x}^j\}_{j=1}^\infty \subseteq I_j$  such that

$$u_{x_1 x_1}(\hat{x}^j) = \frac{u_{x_1}(\bar{x}^j) - u_{x_1}(x_0)}{[\bar{x}^j]_1 - [x_0]} = 0,$$

then by continuity we have

$$u_{x_1 x_1}(x_0) = \lim_{j \rightarrow \infty} u_{x_1 x_1}(\hat{x}^j) = 0. \quad (4.16)$$

If  $f(0) \geq 0$ , then  $\Delta u + c(x)u = \Delta u + f(u) - f(0) \leq 0$  on  $\Omega_\varepsilon$ . By the Hopf Lemma,  $\nabla u(x_0) \cdot w < 0$  for any vector  $w$  such that  $\nu(x_0) \cdot w > 0$ . In particular,  $u_{x_1}(x_0) = \nabla u(x_0) \cdot \gamma < 0$ , a contradiction to equation 4.15.

If  $f(0) < 0$ , then by Lemma 17 we have  $u_{x_i x_j} = [\Delta u(x_0)]\nu_i \nu_j = -f(0)\nu_i \nu_j$  for all  $i$  and  $j$ . As a consequence,  $u_{x_1 x_1}(x_0) = -f(0)\nu_1^2 \neq 0$ , a contradiction to equation 4.16. As a result, the original assumption (that there is no  $\delta \in (0, \varepsilon)$  such that  $u_{x_1} < 0$  on  $\Omega_\delta$ ) is not correct and the lemma is that way proved. ■

**Lemma 19** *Suppose there is a  $\lambda \in [\lambda_1, \lambda_0)$  such that for  $x \in \Sigma(\lambda)$  we have  $u_{x_1}(x) \leq 0$  and  $u(x) \leq u(x^\lambda)$  with  $u(x) \not\equiv u(x^\lambda)$ ; then  $u(x) < u(x^\lambda)$  for  $x \in \Sigma(\lambda)$  and  $u_{x_1}(x) < 0$  for  $x \in \Omega \cap T_\lambda$ .*

**Proof.** For  $\gamma = (1, 0, \dots, 0)$  and  $x \in \Sigma(\lambda)$  note that for  $x^\lambda \in \Sigma'(\lambda)$  is given by  $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ . Let's define  $h(x) := u(x^\lambda)$  for  $x \in \Sigma'(\lambda)[x^\lambda \in \Sigma(\lambda)]$ ; then  $h$  satisfies  $\Delta h + f(h) = 0$  for  $x \in \Sigma'(\lambda)$ . Define  $w(x) := h(x) - u(x)$  for  $x \in \Sigma'(\lambda)$ ; then  $\Delta w + c(x)w = \Delta w + f(h) - f(u) = 0$  for  $x \in \Sigma'(\lambda)$  where  $c(x)$  is constructed using the Mean Value Theorem. Since  $w(x) \leq 0$  for  $x \in \Sigma'(\lambda)$  and  $w(x) = 0$  for  $x \in T_\lambda \cap \Omega$ , by the maximum principle we have  $w(x) < 0$  for  $x \in \Sigma'(\lambda)$ , and by Lemma 16 we obtain  $\frac{\partial w}{\partial x_1} > 0$  for  $x \in T_\lambda \cap \Omega$ .

In that way,  $u(x^\lambda) = h(x) < u(x)$  for  $x \in \Sigma'(\lambda)$  and  $0 < w_{x_1} = h_{x_1} - u_{x_1} = -2u_{x_1}$  for  $x \in T_\lambda \cap \Omega$ . This implies that  $u_{x_1} < 0$  for  $x \in T_\lambda \cap \Omega$  and  $u(x^\lambda) < u(x)$  for  $x \in \Sigma'(\lambda)$ , implying that  $u(x) < u(x^\lambda)$  for  $x \in \Sigma(\lambda)$ . ■

**Lemma 20** *Let  $H(\lambda) = \{x \in \mathbb{R}^n : x_1 > \lambda\}$ . Let  $u(x) > 0$  on  $\Omega$ ,  $u \in C^2(\bar{\Omega} \cap H(\lambda_1))$ , and  $u(x) = 0$  on  $\partial\Omega \cap H(\lambda_1)$ . For any  $\lambda \in (\lambda_1, \lambda_0)$  we have  $u_{x_1}(x) < 0$  and  $u(x) < u(x^\lambda)$  for  $x \in \Sigma(\lambda)$ .*

**Proof.** Because of Lemma 18, for  $\lambda$  close to  $\lambda_0$  with  $\lambda < \lambda_0$  we have

$$u_{x_1}(x) < 0, \text{ and} \tag{4.17}$$

$$u(x) < u(x^\lambda), x \in \Sigma(\lambda).$$

Let's decrease  $\lambda$  until a critical value  $\mu$  is reached,

$$\mu = \inf\{\bar{\lambda} \in [\lambda_1, \lambda_0) : (4.17) \text{ holds for } \lambda \in (\bar{\lambda}, \lambda_0)\}$$

Equation 4.17 holds in this case for  $\mu < \lambda < \lambda_0$  and, for  $\lambda = \mu$ , because of continuity,

$$u_{x_1}(x) \leq 0, \text{ and} \quad (4.18)$$

$$u(x) \leq u(x^\lambda), x \in \Sigma(\mu).$$

We require that  $\mu = \lambda_1$ . Assume not; then  $\mu > \lambda_1$ . For any  $x_0 \in \partial \Sigma(\mu) \setminus T_\mu$  we have  $x_0^\mu \in \Omega$ . Since  $0 = u(x_0) < u(x_0^\mu)$ , (remember  $u > 0$  in  $\Omega$ ) it is seen that  $u(x^\mu) \neq u(x)$  in  $\Sigma(\mu)$  and so Lemma 19 applies. Thus,  $u(x) < u(x^\mu)$  for  $x \in \Sigma(\mu)$  and  $u_{x_1} < 0$  for  $x \in \Omega \cap T_\mu$  and equation (4.17) holds for  $\lambda = \mu$ .

Since  $u_{x_1} < 0$  on  $\Omega \cap T_\mu$ , by Lemma 18 there is an  $\varepsilon > 0$  such that  $u_{x_1} < 0$  on  $\Omega \cap H(\mu - \varepsilon)$ . By the definition of  $\mu$ , there are sequences  $\{\Lambda_j\}_{j=1}^\infty$  and  $\{x_j\}_{j=1}^\infty$  with  $\Lambda_j \in (\mu - \varepsilon, \mu)$  and  $x_j \in \Sigma(\Lambda_j)$  satisfying

$$\Lambda_j \uparrow \mu \text{ as } j \rightarrow \infty \text{ and } u(x_j) \geq u(x_j^{\Lambda_j}) \quad (4.19)$$

By compactness of  $\overline{(\Sigma(\lambda_1))}$  there is a subsequence  $\{x_{j_k}\}_{k=1}^\infty$  such that  $x_{j_k} \rightarrow x \in \Sigma(\mu)$ . Then,

$$x_{j_k}^{\Lambda_{j_k}} \rightarrow x^\mu \in \Sigma(\mu) \text{ and } u(x) \geq u(x^\mu). \quad (4.20)$$

But  $x \in \partial \Sigma(\mu)$  since (4.17) holds for  $\lambda = \mu$ . If  $x \notin T_\mu$ , then  $x^\mu \in \Omega$  and by equation (4.20),  $0 = u(x) \geq u(x^\mu)$  which is a contradiction to  $u > 0$  on  $\Omega$ . In that way,  $x \in T_\mu$  and  $x = x^\mu$ .

For  $k$  sufficiently large, the line segment joining  $x_{j_k}$  and  $x_{j_k}^{\Lambda_{j_k}}$  is in  $\Omega$ . From equation (4.19) and the Mean Value Theorem, there is a  $y_{j_k}$  such that  $u_{x_1}(y_{j_k}) \geq 0$ . Letting  $k \rightarrow \infty$  it is obtained that  $u_{x_1}(x) \geq 0$  where  $x \in T_\mu$ , a contradiction since (4.17) holds for  $\lambda = \mu$ . Thus, our assumption that  $\mu > \lambda_1$  is incorrect. In fact,  $\mu = \lambda_1$  and (4.17) is valid for all  $\lambda \in (\lambda_1, \lambda_0)$ . ■

**Corollary 21** *If  $u_{x_1}(x) = 0$  for some  $x \in \Omega \cap T_{\lambda_1}$ , then  $u$  is symmetric in  $T_{\lambda_1}$  and*

$$\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup [T_{\lambda_1} \cap \Omega].$$

**Proof.** If  $u_{x_1}(x) = 0$  for some  $x \in \Omega \cap T_{\lambda_1}$ , then by Lemma 19 we have  $u(x) \equiv u(x^{\lambda_1})$  for  $x \in \Sigma(\lambda_1)$ . This implies that  $u$  is symmetric relative to  $T_{\lambda_1}$ . Since  $u(x) > 0$  in  $\Sigma(\lambda_1)$  and  $u = 0$  on  $\partial\Omega$ , we conclude that  $\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup [T_{\lambda_1} \cap \Omega]$ . ■

Now, the main result on radial symmetry will be proved

**Theorem 22** For  $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$ , let  $u \in C^2(\bar{\Omega})$  be a positive solution of BVP (4.1) where  $f \in C^1$ ; then  $u = u(r)$  where  $r = |x|$  and  $u'(r) < 0$  for  $r \in (0, R)$ .

**Proof.** By Lemma 20 and Corollary 21,  $u_{x_1} < 0$  for all  $x$  with  $x_1 > 0$ . This implies that  $u_{x_1} > 0$  for  $x_1 < 0$ . As a consequence,  $u_{x_1}(x) = 0$  for  $x_1 = 0$ . By Corollary 21,  $u$  is symmetric in  $x_1$ . Since the direction vector  $\gamma$  is arbitrary, the argument above works for any direction. It follows that  $u$  is radially symmetric and  $u_r < 0$  for  $0 < r < R$ . ■



## Appendix A

# Limits, Inferior Limits and Superior Limits

In this section  $\overline{\mathbb{R}}$  will denote the set  $\mathbb{R} \cup \{+\infty, -\infty\}$  of extended reals.  $\overline{\mathbb{R}}$  is endowed with a metric  $D$  defined as follows.

$$D(x, y) = |\mu(x) - \mu(y)| \quad (\text{A.1})$$

where

$$\mu(x) = \begin{cases} \frac{x}{1+|x|} & \text{if } x \in \mathbb{R} \\ 1 & \text{if } x = +\infty \\ -1 & \text{if } x = -\infty \end{cases} \quad (\text{A.2})$$

It is easy to verify that the function  $\mu : (\overline{\mathbb{R}}, D) \rightarrow ([-1, 1], d)$  where  $d(x, y) = |x - y|$ , is an isometry of metric spaces. Also, notice that every  $A \subset \overline{\mathbb{R}}$  has a supremum and an infimum. A *sequence of extended real numbers* will mean a function  $f : \mathbb{N} \rightarrow \overline{\mathbb{R}}$ .

**Definition 23** *Let  $f$  be a sequence of extended real numbers.*

1. *We say that  $f$  converges to  $L \in \overline{\mathbb{R}}$  if  $D(f(n), L)$  converges to zero in the usual sense. In this case,  $L$  is called the limit of  $f$  and is denoted by  $\lim_{n \rightarrow \infty} f(n)$ .*
2. *We define the inferior limit of  $f$ , as*

$$\liminf_{n \rightarrow \infty} f(n) := \sup \{ \inf \{ f(l) : l \geq k \} : k \in \mathbb{N} \},$$

3. *and the superior limit of  $f$ , as*

$$\limsup_{n \rightarrow \infty} f(n) := \inf \{ \sup \{ f(l) : l \geq k \} : k \in \mathbb{N} \}$$

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Notice that  $h(k) = \inf \{ f(l) : l \geq k \}$  is also a sequence of extended real numbers, and that it is nondecreasing, that is,  $h(k_1) \leq h(k_2)$  whenever  $k_1 < k_2$ . (This is due to the fact that if  $A \subset B \subset \overline{\mathbb{R}}$  then  $\inf B \leq \inf A$ .) Therefore  $\lim_{n \rightarrow \infty} h(n)$  always exists and coincides with  $\sup \{ h(n) : n \in \mathbb{N} \}$ . The same is true for the superior limit. It is also the case that if the inferior and superior limits of  $f$  coincide and equal an extended real  $L$ , then  $f$  converges to  $L$ .

Now, if one replaces  $\mathbb{N}$  by the interval  $(0, \infty)$ , one obtains not only a generalization of the concept of sequence of extended real numbers but also of its convergence, inferior limit and superior limit.

**Definition 24** Let  $f : (0, \infty) \rightarrow \overline{\mathbb{R}}$  be a function.

1.  $f$  is said to converge to  $L \in \overline{\mathbb{R}}$  if  $D(f(s), L)$  converges to zero as  $s$  goes to  $+\infty$ . In this case the extended real number  $L$  is called the limit of  $f$  when  $s$  goes to infinity, and is denoted by  $\lim_{s \rightarrow \infty} f(s)$ .

2. We define the inferior limit of  $f$  as

$$\liminf_{s \rightarrow \infty} f(s) := \sup \{ \inf \{ f(t) : t \geq s \} : s \in (0, \infty) \},$$

3. and the superior limit of  $f$  as

$$\limsup_{s \rightarrow \infty} f(s) := \inf \{ \sup \{ f(t) : t \geq s \} : s \in (0, \infty) \}.$$

Notice that the function  $g : (0, \infty) \rightarrow \overline{\mathbb{R}}$  defined by  $g(s) = \inf \{ f(t) : t \geq s \}$  is defined at each  $s \in (0, \infty)$  and is a nondecreasing function, that is,  $g(s_1) \leq g(s_2)$  whenever  $s_1 < s_2$ . This implies that the inferior limit always exists (in  $\overline{\mathbb{R}}$ ). For similar reasons, the superior limit always exists. It is also true that if the inferior and superior limits coincide and equal an extended real number  $L$ , then  $f$  converges to  $L$ .

Let  $X$  be a set and  $Map(X, \overline{\mathbb{R}})$  denote the collection of all functions from  $X$  into  $\overline{\mathbb{R}}$ . A *sequence of real-valued functions on  $X$*  is just a function  $F : \mathbb{N} \rightarrow Map(X, \overline{\mathbb{R}})$ . In this case we can define the notions of limit, inferior limit and superior limit, as follows.

**Definition 25**

1. The sequence of functions  $F$  is said to converge to the function  $h : X \rightarrow \overline{\mathbb{R}}$  if for each  $x \in X$ , the sequence of extended real numbers  $F(\cdot)(x) : \mathbb{N} \rightarrow \overline{\mathbb{R}}$  converges to  $h(x)$ . When such an  $h$  exists, it is unique, and it is called the pointwise limit of the sequence of functions  $F$ . It is usually denoted by  $\lim_{n \rightarrow \infty} F(n)$ . Furthermore, the sequence  $F$  is said to converge uniformly to  $h$  if for each  $\epsilon > 0$  there is an  $N(\epsilon) \in \mathbb{N}$  such that if  $n \geq N(\epsilon)$  then  $D(F(n)(x), h(x)) < \epsilon$  for every  $x \in X$ .
2. The inferior limit of  $F$  is the function  $\liminf_{n \rightarrow \infty} F(n) : X \rightarrow \overline{\mathbb{R}}$  sending each  $x \in X$  to  $\liminf_{n \rightarrow \infty} F(n)(x)$ . In words, it sends each  $x \in X$  to the inferior limit of the sequence of reals  $F(n)(x)$ . Equivalently,  $\liminf_{n \rightarrow \infty} F(n)$  can be defined as  $\lim_{n \rightarrow \infty} h(n)$  where for each  $n$ ,  $h(n) : X \rightarrow \overline{\mathbb{R}}$  is defined as  $h(n)(x) = \inf\{F(k)(x) : k \geq n\}$ .  $F$  is said to approach  $\liminf_{n \rightarrow \infty} F(n)$  uniformly when the sequence of functions  $h(n)$  converges uniformly to its limit  $\lim_{n \rightarrow \infty} h(n)$ .
3. The superior limit of  $F$  is the function  $\limsup_{n \rightarrow \infty} F(n) : X \rightarrow \overline{\mathbb{R}}$  sending each  $x \in X$  to  $\limsup_{n \rightarrow \infty} F(n)(x)$ . In words, it sends each  $x \in X$  to the superior limit of the sequence of reals  $F(n)(x)$ . Equivalently,  $\limsup_{n \rightarrow \infty} F(n)$  can be defined as  $\lim_{n \rightarrow \infty} l(n)$  where for each  $n$ ,  $l(n) : X \rightarrow \overline{\mathbb{R}}$  is defined as  $l(n)(x) = \sup\{F(k)(x) : k \geq n\}$ .  $F$  is said to approach  $\limsup_{n \rightarrow \infty} F(n)$  uniformly when the sequence of functions  $l(n)$  converges uniformly to its limit  $\lim_{n \rightarrow \infty} l(n)$ .

Notice that the sequence of functions  $h(n)$  defined in 2 is nondecreasing, that is, if  $n_1 < n_2$  then  $h(n_1)(x) \leq h(n_2)(x)$  for every  $x \in X$ . This implies that  $\lim_{n \rightarrow \infty} h(n)$  always exists. A similar remark applies to the sequence  $l(n)$  where one replaces the word nondecreasing by nonincreasing.

Since specifying a sequence of real-valued functions  $F : \mathbb{N} \rightarrow \text{Map}(X, \overline{\mathbb{R}})$  is the same as specifying a family  $\{f_n\}_{n \in \mathbb{N}}$ , where each  $f_n$  is a function from  $X$  to  $\overline{\mathbb{R}}$ , sequences of real-valued functions will be denoted by  $\{f_n\}_{n \in \mathbb{N}}$ ,  $\{g_n\}_{n \in \mathbb{N}}$ , etc.

The following basic result is useful in many situations.

**Theorem 26** *Let  $(X, d)$  be a metric space, and let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of continuous real-valued functions on  $X$ . If  $f := \lim_{n \rightarrow \infty} f_n$  exists and  $\{f_n\}_{n \in \mathbb{N}}$  converges uniformly to  $f$ , then  $f$  is continuous.*

**Proof.** Let  $x_0$  be a point in  $X$ , and let  $\epsilon$  be a positive real number. There exist  $\delta > 0$  and  $N$  such that if  $d(x, x_0) < \delta$  then

1.  $d(f(x), f_N(x)) < \epsilon/3$ ,
2.  $d(f_N(x), f_N(x_0)) < \epsilon/3$ ,
3.  $d(f_N(x_0), f(x_0)) < \epsilon/3$ ,

and therefore

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

■

If one replaces  $\mathbb{N}$  by the interval  $(0, \infty)$  one obtains the notion of *uniparametric family of functions* and of its convergence and limit, inferior limit and superior limit.

**Definition 27** An uniparametric family of real-valued functions on  $X$  is a function  $F : (0, \infty) \rightarrow \text{Map}(X, \overline{\mathbb{R}})$ . For each  $s > 0$ , the function from  $X$  to  $\overline{\mathbb{R}}$  sending  $x \in X$  to  $F(s)(x)$  will be denoted by  $F_s$  and to the entire family by  $\{F_s\}_{s \in (0, \infty)}$ .

From here on we will denote uniparametric families by  $\{\phi_s\}_{s \in (0, \infty)}$ ,  $\{\psi_s\}_{s \in (0, \infty)}$ , etc.

**Definition 28** Let  $\{\phi_s\}_{s \in (0, \infty)}$  be an uniparametric family of extended real-valued functions on  $X$ .

1.  $\{\phi_s\}_{s \in (0, \infty)}$  is said to converge to the function  $h : X \rightarrow \overline{\mathbb{R}}$  if for each  $x \in X$ , the function  $\phi_s(x) : (0, \infty) \rightarrow \overline{\mathbb{R}}$  converges to  $h(x)$ . When such an  $h$  exists, it is unique, and it is called the pointwise limit of the family  $\{\phi_s\}_{s \in (0, \infty)}$ . It is usually denoted by  $\lim_{s \rightarrow \infty} \phi_s$ . Furthermore, the family  $\{\phi_s\}_{s \in (0, \infty)}$  is said to converge uniformly to  $h$  if for each  $\epsilon > 0$  there is an  $S(\epsilon) \in (0, \infty)$  such that if  $s \geq S(\epsilon)$  then  $D(\phi_s(x), h(x)) < \epsilon$  for every  $x \in X$ .
2. The inferior limit of  $\{\phi_s\}_{s \in (0, \infty)}$  is the function  $\liminf_{s \rightarrow \infty} \phi_s : X \rightarrow \overline{\mathbb{R}}$  sending each  $x \in X$  to  $\liminf_{s \rightarrow \infty} \phi_s(x)$ . In words, it sends each  $x \in X$  to the inferior limit of the function  $\phi_s(x) : (0, \infty) \rightarrow \overline{\mathbb{R}}$ .

3. The superior limit of  $\{\phi_s\}_{s \in (0, \infty)}$  is the function  $\limsup_{s \rightarrow \infty} \phi_s : X \rightarrow \overline{\mathbb{R}}$  sending each  $x \in X$  to  $\limsup_{s \rightarrow \infty} \phi_s(x)$ . In words, it sends each  $x \in X$  to the superior limit of the function  $\phi_s(x) : (0, \infty) \rightarrow \overline{\mathbb{R}}$ .

We also have in this case the following basic result.

**Theorem 29** Let  $(X, d)$  be a metric space, and let  $\{\phi_s\}_{s \in (0, \infty)}$  be an uniparametric family of continuous real-valued functions on  $X$ . If  $\psi := \lim_{s \rightarrow \infty} \phi_s$  exists and  $\{\phi_s\}_{s \in (0, \infty)}$  converges uniformly to  $\psi$ , then  $\psi$  is continuous.

**Proof.** Let  $x_0$  be a point in  $X$ , and let  $\epsilon$  be a positive real number. There exist  $\delta > 0$  and  $S > 0$  such that if  $d(x, x_0) < \delta$  then

1.  $d(\psi(x), \phi_S(x)) < \epsilon/3$ ,
2.  $d(\phi_S(x), \phi_S(x_0)) < \epsilon/3$ ,
3.  $d(\phi_S(x_0), \psi(x_0)) < \epsilon/3$ ,

and therefore

$$\begin{aligned} d(\psi(x), \psi(x_0)) &\leq d(\psi(x), \phi_S(x)) + d(\phi_S(x), \phi_S(x_0)) + d(\phi_S(x_0), \psi(x_0)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

■

Let us see that the inferior limit of an uniparametric family is always the pointwise limit of another uniparametric family which is nondecreasing. More precisely, let  $\{\phi_s\}_{s \in (0, \infty)}$  be an uniparametric family and let  $h$  denote its inferior limit. Now for each  $s \in (0, \infty)$  let us define a function  $\psi_s : X \rightarrow \overline{\mathbb{R}}$  by

$$\psi_s(x) = \inf \{\phi_t(x) : t \geq s\}.$$

The family  $\{\psi_s\}_{s \in (0, \infty)}$  is *monotone increasing* in the following precise sense: if  $0 \leq s_1 < s_2$  then  $\psi_{s_1}(x) \leq \psi_{s_2}(x)$  for each  $x \in X$ . This immediately follows since  $\{\phi_t(x) : t \geq s_2\} \subset \{\phi_t(x) : t \geq s_1\}$  and  $\inf A \leq \inf B$  whenever  $B \subset A \subset \overline{\mathbb{R}}$ .

The following lemma will be used in proving Theorem 31. Let  $(X, d)$  be a metric space. Remember that a family of functions  $\{f_\alpha : \alpha \in A\}$  where each  $f_\alpha$  is a function from  $X$  to  $\mathbb{R}$ , is said to be *equicontinuous* if for every  $\epsilon > 0$  there is a  $\delta(\epsilon) > 0$  such that for each  $\alpha \in A$ ,  $|f_\alpha(x) - f_\alpha(y)| < \epsilon$  whenever  $d(x, y) < \delta(\epsilon)$ .

**Lemma 30** *Let  $(X, d)$  be a metric spaces, and let  $\{f_\alpha : \alpha \in A\}$  be an equicontinuous family of functions from  $X$  to  $[0, \infty)$ . Then the function  $g : X \rightarrow [0, \infty)$  defined by  $g(x) = \inf\{f_\alpha(x) : \alpha \in A\}$ , is uniformly continuous, and therefore continuous.*

**Proof.** Let us fix  $\epsilon > 0$ . There is a number  $\delta > 0$  such that for each  $\alpha \in A$ ,  $|f_\alpha(x) - f_\alpha(y)| < \epsilon/2$  whenever  $d(x, y) < \delta$ . There exists an index  $\alpha'$  such that  $|f_{\alpha'}(x) - g(x)| < \epsilon/2$ . Since  $|f_{\alpha'}(x) - f_{\alpha'}(y)| < \epsilon/2$ , the triangle inequality tells us that

$$|g(x) - f_{\alpha'}(y)| < \epsilon \tag{A.3}$$

On the other hand, we know that

$$g(x) \leq f_{\alpha'}(y) \tag{A.4}$$

Combining inequalities (A.3) and (A.4) we obtain the inequality

$$g(y) \leq g(x) + \epsilon. \tag{A.5}$$

Interchanging the roles of  $x$  and  $y$  we obtain

$$g(x) \leq g(y) + \epsilon. \tag{A.6}$$

But the last two inequalities imply that  $|g(x) - g(y)| \leq \epsilon$ . This proves that  $g$  is uniformly continuous, and therefore that it is continuous. ■

**Theorem 31** *Let  $s_0$  be a positive real number and let  $\{\phi_s\}_{s \in [s_0, \infty)}$  be an uniparametric family of  $[0, \infty)$ -valued functions on a compact metric space  $(X, d)$ . Assume that the family is equicontinuous and that it approaches uniformly the function  $\psi := \liminf_{s \rightarrow \infty} \phi_s$ . If  $\psi(x) > 0$  for every  $x \in X$ , then there is a  $\beta > 0$  and a  $s_1 \geq s_0$  such that  $\phi_s(x) > \beta$  for every  $s \geq s_1$  and  $x \in X$ .*

**Proof.** For each  $s \in [s_0, \infty)$  define the function  $\psi_s : X \rightarrow [0, \infty)$  as  $\psi_s(x) = \inf\{\phi_t(x) : t \geq s\}$ . Since the family  $\{\phi_t\}_{t \in [s, \infty)}$  is equicontinuous, Lemma 30 implies that each  $\psi_s$  is a continuous function. Since the family of (continuous) functions  $\psi_s$  converges uniformly to  $\psi$ , we have that  $\psi$  is also a continuous function. This fact, together with the compactness assumption on  $X$ , shows that  $\psi$  attains an absolute minimum value  $m$ . The number  $m$  must be strictly positive because we have assumed that  $\psi(x) > 0$  for every  $x \in X$ . The fact that the family  $\{\psi_s\}_{s \in [s_0, \infty)}$  is nondecreasing (i.e. if  $s_1 < s_2$  then  $\psi_{s_1}(x) \leq \psi_{s_2}(x)$ ) and approaches  $\psi$  uniformly implies that there exists a  $s_1 \in [s_0, \infty)$  such that if  $s \geq s_1$  then  $|\psi_s(x) - \psi(x)| < \frac{m}{2}$  for every

$x \in X$ . In particular this implies that  $-\frac{m}{2} < \psi_s(x) - \psi(x)$  for each  $x \in X$ , and therefore that  $\psi(x) - \frac{m}{2} < \psi_s(x)$  for each  $x \in X$ . But we have that  $m \leq \psi(x)$  for each  $x \in X$ . We conclude that for each  $s \geq s_1$ ,  $\frac{m}{2} < \psi_s(x)$  for every  $x \in X$ . This means that  $\frac{m}{2} < \inf\{\phi_t(x) : t \geq s\}$  for every  $x \in X$ , which in turn implies that for each  $t \geq s_1$ ,  $\frac{m}{2} < \phi_t(x)$  for every  $x \in X$ . ■



## Appendix B

# The Interior Ball Condition

**Definition 32** Let  $\Omega$  be an open set in  $\mathbb{R}^n$ . Then  $\Omega$  is said to satisfy the interior ball condition at  $x \in \partial\Omega$  if there exists an open ball  $B \subset \Omega$  such that  $x \in \overline{B}$ .

**Definition 33** An open set  $\Omega \subset \mathbb{R}^n$  is said to be of class  $C^2$  if its boundary  $\partial\Omega$  is an  $(n-1)$ -dimensional manifold of class  $C^2$ , that is, for each  $x = (x_1, \dots, x_n) \in \partial\Omega$  there exist:

1. an open set  $N_x \subset \mathbb{R}^n$  containing  $x$ ,
2. an index  $1 \leq i \leq n$ ,
3. an open set  $U_x \subset \mathbb{R}^{n-1}$  containing the point  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  and
4. a function  $h_x : U_x \rightarrow \mathbb{R}$  in  $C^2(U)$  (see section C.1 for the definition),

such that

$$\partial\Omega \cap N_x = \{(z_1, \dots, z_{i-1}, h_x(z), z_i, \dots, z_{n-1}) : z = (z_1, \dots, z_{n-1}) \in U_x\} \quad (\text{B.1})$$

**Theorem 34** If  $\Omega$  is of class  $C^2$  then it satisfies the interior ball condition at each  $x \in \partial\Omega$ .

**Proof.**

■ Let  $f(x) = ax^2 + \phi(x)$  be a  $C^2$  function, because of Taylor formula can be said that  $\|\frac{\phi(x)}{x^2}\| \rightarrow 0$  as  $x \rightarrow 0$ . For  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $\|\frac{\phi(x)}{x^2}\| < \varepsilon$  if  $-\delta < x < \delta$  or  $\|\phi(x)\| < \varepsilon x^2$  if  $-\delta < x < \delta$ .

In two variables  $f(x, y) = ax^2 + by^2 + cxy + \phi(x, y)$  of class  $C^2$  and therefore  $\frac{\|\phi(x, y)\|}{\|(x, y)\|^2} \rightarrow 0$  as  $(x, y) \rightarrow 0$  then  $\|\phi(x, y)\| \leq \varepsilon(x^2 + y^2) = \varepsilon x^2 + \varepsilon y^2$ .

In general, Taylors theorem establishes that for  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  with continuous partial derivatives up to the third order, then we can write

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2} \sum_{i,j=1}^n h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) + R_2(\mathbf{h}, \mathbf{x}_0) \quad (\text{B.2})$$

where  $R_2(\mathbf{h}, \mathbf{x}_0)/\|\mathbf{h}\|^2 \rightarrow 0$  as  $\mathbf{h} \rightarrow \mathbf{0}$ .

**Definition 35** Unit Outside Normal Vector *Let  $\Omega$  be an open set in  $\mathbb{R}^N$  and let  $x^0 \in \partial\Omega$ , let's suppose there exists an open ball  $B$  such that  $x^0 \in \partial B$  and  $B \subset \Omega$ . Then we call the unit outside normal vector of  $\Omega$  in  $x^0$  to the vector  $\frac{\overrightarrow{ox^0}}{\|\overrightarrow{ox^0}\|}$  where  $o$  is the center of  $B$  and we denote it by  $\nu(x^0)$ . When  $\partial\Omega$  is  $C^2$  class, then in a point  $x^0 \in \partial\Omega$  the interior ball property condition is satisfied and  $\nu(x^0)$  can be defined as above.*

## Appendix C

# Function Spaces

Let us briefly recall the multiindex notation for partial derivatives of functions. Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and let  $f : \Omega \rightarrow \mathbb{R}$  be a function. A *multiindex* is an  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n)$  of nonnegative integers and its *order* is defined as  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . Given a multiindex  $\alpha$ ,  $D^\alpha f(x)$  denotes the partial derivative

$$\frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

for each  $x \in \Omega$ . We observe that the multiindex notation is generally used only on functions for which the differentiation order is unimportant.

### C.1 $C^k(\Omega), C^k(\overline{\Omega})$

Let  $k \geq 0$  be an integer. A function  $f : \Omega \rightarrow \mathbb{R}$  is said to be of class  $C^k$  in  $\Omega$ , or equivalently, to belong to  $C^k(\Omega)$  if for every multiindex  $\alpha$  with  $|\alpha| \leq k$ , the function  $D^\alpha f : \Omega \rightarrow \mathbb{R}$  is defined and continuous at every  $x \in \Omega$ . It is also important to consider the space of function defined as follows. A function  $f : \overline{\Omega} \rightarrow \mathbb{R}$  is said to belong to the class  $C^k(\overline{\Omega})$  if  $f|_\Omega \in C^k(\Omega)$  and for each  $\alpha$  with  $|\alpha| \leq k$ , the function  $D^\alpha(f|_\Omega)$  is uniformly continuous on each bounded subset of  $\Omega$ , i.e. for each bounded set  $A \subset \Omega$  and each  $\epsilon > 0$  there exists a  $\delta(\epsilon) > 0$  such that  $|D^\alpha f(x) - D^\alpha f(y)| < \epsilon$  whenever  $\|x - y\| < \delta(\epsilon)$ , for every  $x, y \in A$ .

It is clear that if  $k > k'$  then  $C^k(\Omega) \subset C^{k'}(\Omega)$  and  $C^k(\overline{\Omega}) \subset C^{k'}(\overline{\Omega})$ .

There is an alternative way to describe the  $C^k(\overline{\Omega})$  spaces which is important to be aware of. It is an immediate consequence of the following

lemma.

**Lemma 36** *Let  $\Omega \subset \mathbb{R}^n$  be an open set, and let  $g : \Omega \rightarrow \mathbb{R}$  be a continuous function. Then there exists a continuous function  $\bar{g} : \bar{\Omega} \rightarrow \mathbb{R}$  such that  $\bar{g}(x) = g(x)$  for every  $x \in \Omega$  if and only if  $g$  is uniformly continuous in every bounded subset of  $\Omega$ . Furthermore, such extension is unique.*

**Proof.** Let us assume that such  $\bar{g}$  exists. Let  $A \subset \Omega$  be a bounded set. Since  $\bar{A} \subset \bar{\Omega}$ , the function  $\bar{g}$  is defined and continuous on  $\bar{A}$ . Now,  $\bar{A}$  is closed and bounded, and therefore it is compact. This, combined with the standard fact that a continuous function defined on a compact set is also uniformly continuous, implies that  $g|_A = \bar{g}|_A$  is uniformly continuous.

Conversely, let us assume that  $g : \Omega \rightarrow \mathbb{R}$  is uniformly continuous when restricted to each bounded subset of  $\Omega$ . Let  $x_0$  be a point in  $\partial\Omega$ . Then there exists a sequence  $\{x_n\}$  of points in  $\Omega$  converging to  $x_0$ . Let us see that the sequence of real numbers  $\{g(x_n)\}$  satisfies the Cauchy condition. Let us fix an  $\epsilon > 0$ . Since the set  $A = \{x_n : n \geq 1\}$  is a bounded subset of  $\Omega$  and the restriction of  $g$  to  $A$  is uniformly continuous, there exists a number  $\delta > 0$  such that if  $\|x - y\| < \delta$  then  $|g(x) - g(y)| < \epsilon$  for each  $x, y \in A$ . Now, the convergence of  $\{x_n\}$  implies that there exists  $N$  such that  $\|x_m - x_n\| < \delta$  whenever  $m, n \geq N$ . So, if  $m, n \geq N$  we have that  $|g(x_m) - g(x_n)| < \epsilon$ .

Having confirmed that  $\{g(x_n)\}$  is a sequence of real numbers satisfying the Cauchy condition, the completeness of the reals guarantees that its limit exists. Let us denote this limit by  $L_{x_0}$ . This limit is independent of the particular sequence  $\{x_n\}$  we chose. In fact, suppose we had chosen another sequence  $\{y_n\}$  converging to  $x_0$ . The uniform continuity of the restriction of  $g$  to the bounded set  $\{x_n : n \geq 1\} \cup \{y_n : n \geq 1\}$ , combined with the fact that  $\|x_n - y_n\|$  approaches zero as  $n$  goes to infinity, tells us that  $|g(x_n) - g(y_n)|$  also approaches zero as  $n$  goes to infinity. This implies that the limits of  $\{g(x_n)\}$  and  $\{g(y_n)\}$  coincide. Let us define the function  $\bar{g} : \bar{\Omega} \rightarrow \mathbb{R}$  as  $\bar{g}(x) = g(x)$  if  $x \in \Omega$  and  $\bar{g}(x) = L_x$  if  $x \in \partial\Omega$ .

We claim that  $\bar{g}$  is continuous on  $\bar{\Omega}$ . Since  $\bar{g}$  coincides with  $g$  in the open set  $\Omega$  it immediately follows that  $\bar{g}$  is continuous at every point in  $\Omega$ . Let  $x_0$  be a point in  $\partial\Omega$  and let  $\{z_n\}$  be a sequence in  $\bar{\Omega}$  converging to  $x_0$ . The construction of  $\bar{g}$  and the fact that each  $z_n$  belongs to  $\bar{\Omega}$ , guarantee that for each  $n$  there exists a point  $x_n \in \Omega$  such that i)  $\|x_n - z_n\| < 1/n$  and ii)  $|\bar{g}(x_n) - \bar{g}(z_n)| < 1/n$ . Now, i) implies that  $\{x_n\}$  converges to  $x_0$ , making  $\{\bar{g}(x_n)\} = \{g(x_n)\}$  converge to  $L_{x_0} = \bar{g}(x_0)$ . On the other hand, ii) implies

that  $\{\bar{g}(z_n)\}$  has the same limit as  $\{\bar{g}(x_n)\}$ , namely  $\bar{g}(x_0)$ . This confirms the continuity of  $\bar{g}$  at  $x_0$ . We conclude that  $\bar{g}$  is continuous on  $\bar{\Omega}$  as claimed.

Finally, let  $h : \Omega \rightarrow \mathbb{R}$  be a continuous function admitting continuous extensions  $\bar{h}_1, \bar{h}_2 : \bar{\Omega} \rightarrow \mathbb{R}$ . Let us see that these functions are necessarily equal. If  $x \in \Omega$  then  $\bar{h}_1(x) = \bar{h}_2(x)$  because they are both extensions of  $h$ . If  $x \in \partial\Omega$  then there exists a sequence  $\{x_n\}$  of points in  $\Omega$  converging to  $x$ . Now, the continuity of  $\bar{h}_1$  and  $\bar{h}_2$  implies that the sequences  $\{\bar{h}_1(x_n)\}$  and  $\{\bar{h}_2(x_n)\}$  converge to  $\bar{h}_1(x)$  and to  $\bar{h}_2(x)$ , respectively. But since both sequences are the same as the sequence  $\{h(x_n)\}$ , we conclude that  $\bar{h}_1(x) = \bar{h}_2(x)$ . ■

As a consequence, the space  $C^k(\bar{\Omega})$  can be defined as the set of functions  $f : \bar{\Omega} \rightarrow \mathbb{R}$  such that  $f|_{\Omega} \in C^k(\Omega)$  and for each multiindex  $\alpha$  with  $|\alpha| \leq k$ , the function  $D^\alpha f : \Omega \rightarrow \mathbb{R}$  admits a continuous extension  $\overline{D^\alpha f} : \bar{\Omega} \rightarrow \mathbb{R}$ . In this case, the function  $\overline{D^\alpha f}$  is usually denoted by  $D^\alpha f$ .

## C.2 Hölder Spaces

Let  $\Omega$  be an open set in  $\mathbb{R}^n$ , and  $0 < \gamma \leq 1$ . A function  $f : \Omega \rightarrow \mathbb{R}$  is said to be *Lipschitz continuous* if there exists a positive constant  $C$  such that

$$|f(x) - f(y)| \leq C\|x - y\| \quad (\text{C.1})$$

for every  $x, y \in \Omega$ . Notice that this condition implies that  $f$  is continuous. Now  $f$  is said to be *Hölder continuous with exponent  $\gamma$*  if there is a positive constant  $C$  such that  $f$  satisfies the inequality

$$|f(x) - f(y)| \leq C\|x - y\|^\gamma \quad (\text{C.2})$$

for every  $x, y \in \Omega$ . Let  $k \geq 0$  be an integer. The *Hölder space*  $C^{k,\gamma}(\bar{\Omega})$  is defined as the set of functions  $f \in C^k(\bar{\Omega})$  all of whose  $k^{\text{th}}$  order partial derivatives are bounded and Hölder continuous with exponent  $\gamma$ .



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