Bifurcations and Sequences of Elements in Nonsmooth Systems Cycles

Ivan Arango [(1)], Fabio Pineda [(1)], Oscar Ruiz [(2)]
(1) Mechatronics and Machine Design Group, Universidad EAFIT, Medellin, Colombia
(2) Laboratorio de CAD/CAM/CAE, Universidad EAFIT, Medellin, Colombia

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Abstract

This article describes the implementation of a novel method for detection and continuation of bifurcations in non-smooth complex dynamic systems. The method is an alternative to existing ones for the follow-up of associated phenomena, precisely in the circumstances in which the traditional ones have limitations (simultaneous impact, Filippov and first derivative discontinuities and multiple discontinuous boundaries). The topology of cycles in non-smooth systems is determined by a group of ordered segments and points of different regions and their boundaries. In this article we compare the limit cycles of non-smooth systems against the sequences of elements, in order to find patterns. To achieve this goal, a method was used, which characterizes and records the elements comprising the cycles in the order that they appear during the integration process. The characterization discriminates: (a) types of points and segments, (b) direction of sliding segments, and (c) regions or discontinuity boundaries to which each element belongs. When a change takes place in the value of a parameter of a system, our comparison method is an alternative to determine topological changes and hence bifurcations and associated phenomena. This comparison has been tested in systems with discontinuities of three types: (1) impact, (2) Filippov and (3) first derivative discontinuities. By coding well known cycles as sequences of elements, an initial comparison database was built. Our comparison method offers a convenient approach for large systems with more than two regions and more than two sliding segments.

Keywords

bifurcation sequences, non-smooth systems, limit cycles, dynamic systems

1 Introduction

Physical systems can often operate in different modes, and due to the time of the transition from one mode to another mode is small, the transition is considered as instantaneous [1]. Events such as impact, dry friction, backlash, hysteresis, saturation and commutation carry a discontinuity or sudden change. Therefore, they can be modeled declaring at least two modes. Each mode is represented by differential equation or mixes of differential and difference equations. The mathematical modeling of these systems switches between different modes and they are classified as piecewise-smooth or nonsmooth system.

Piecewise-smooth systems may be classified according to the degree of discontinuity that the orbits and vector fields present [1]. An updated classification by [2] discusses systems with three degrees of smoothness. In the zero level, one has jumps in the state variables. They are typically systems with impact, where the phenomenon is modeled assuming no deformation and a negligible impact time [3]. In the first degree of smoothness we have systems described by differential equations with discontinuous right hand terms (Filippov systems) [4]. In these cases the vector field is discontinuous in the switching Boundary, as usual in mechanical systems with dry friction [5]. The second degree of smoothness, includes systems with continuous vector fields but discontinuities in the first derivative of the vector field. As an example for second degree, we might consider a mechanical system with a single mass, spring, damping element and limiting elastic support [6]. In general, a discontinuity in the i-th derivative implies that the system is classified as being i+1 degree of smoothness.

Non-standard bifurcations in nonsmooth systems have been intensively studied [6],[7], [8], [9]. But, there are only mathematical tools to analyze phenomena in 2D or 3D systems with two vector fields and one discontinuity boundary [10],[11]. The names assigned to the bifurcations vary according to the researcher. For example, in [2] is used *Grazing*, *Switching*, *Crossing* and *Multisliding*. For the same bifurcations, in [12] is used *Touching*, *Bucking*, *Crossing* and *Adding*. Other sliding bifurcation types, recently reported in [8], have been characterized in systems with two DBs. Those bifurcations have been called *Exchanging*, *Sticking Disappearance* and *Nonsmooth Fold*.

Article Outline. This article is organized as follows. Section 2 explains the notation and symbols used. Sec-

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tion 3 summarizes the solutions for the types of nonsmooth systems. Section 4 describes the well known bifurcations as sequences of elements. Section 5 analyzes the procedure of identification and comparison of the elements of the cycles versus the elements of an integration. Section 6 concludes the article.

2 Notation and Symbology for Points in the DB

The study of nonsmooth systems includes more information than a smooth system. The proposed method is based on the information of each element of the cycle. Therefore, we had to introduce a notation to see all the information of the points, segments and orbits. The information should be fully contained inside the textual or graphical symbols assigned to each element. Some distinguished symbols follow.

$\mathbf{x}$	:	State variable vector, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
Z_i	:	i-th smooth region of the space state.
α	:	Parameter of the physical system ($\alpha \in \mathbb{R}$).
$\mathbf{F}_i(\mathbf{x}, \alpha)$	:	Vector field on region Z_i.
DB	:	Discontinuity Boundary.
Σ_{ij}	:	Discontinuity Boundary between regions Z_i and Z_j. $\Sigma_{ij} = \bar{Z}_i \cap \bar{Z}_j = \{\mathbf{x} \in \mathbb{R}^n : H_{ij}(\mathbf{x}, \alpha) = 0\}$.
$H_{ij}(\mathbf{x}, \alpha)$	:	Smooth scalar function defining the DB between regions i and j. $H_{ij}(\mathbf{x}, \alpha) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$.
$H_{ij_{\mathbf{x}}}(\mathbf{x}, \alpha)$	:	Gradient of $H_{ij}(\mathbf{x}, \alpha)$. $H_{ij_{\mathbf{x}}}(\mathbf{x}, \alpha) = \left(\frac{\partial H_{ij}(\mathbf{x},\alpha)}{\partial x_1}, \ldots, \frac{\partial H_{ij}(\mathbf{x},\alpha)}{\partial x_n}\right)$.
Ω_{Ii}^{-}	:	i-th component of $\mathbf{x}$ before impact.
Ω_{Ii}^{+}	:	i-th component of $\mathbf{x}$ after impact.
γ	:	Impact restitution coefficient $\gamma = \lvert\dot{\Omega}_I^{-}/\dot{\Omega}_I^{+}\rvert$.
$\mathbf{x}_i$	:	Point at the end of i-th integration step.
$\mathbf{G}_{ij}(\mathbf{x}, \alpha)$	:	Vector field that acts on the DB between regions i and j, for sliding.

Cycle equations include indicators, separators and elements (for cycles: points or segments). Cycles are identified with a letter C accompanied by a subscript number (e.g. C_4: 4-th cycle). If the cycle contains sliding segments they appear as S superscript preceding the C letter (e.g. ${}^{S}C_5$: cycle 5 has sliding segments). In the equations, the symbol Φ is used to represent a composed segment, determined by a sequence of points of a common type (e.g. Φ_5: a composed segment in region 5). The points are identified with the letter Ω with superindices (- or +) indicating whether the point is an initial (-) or endpoint (+) of a sliding segment S. The symbol / notes a separator between consecutive elements.

The indicator $\circlearrowright$ shows that the elements of the equation in an evolution are continuously repeated (e.g. $\Phi_i/\circlearrowright$: segment Φ_i in region i is continuously repeated). Equations that describe the elements of Bifurcations (cycles) are identified by the symbol β. Sliding bifurcations are identified with a super-script S that precedes the β symbol and an alphabetic sub-script that indicates the bifurcation type (e.g. ${}^{S}\beta_c$ is a sliding crossing bifurcation).

3 Background of the nonsmooth solution

Typically, nonsmooth systems are modeled as piecewise-smooth systems (PWS) where the state space contains four kinds of spaces: *Smooth Zones*, *undefined Zones* associated to regions behind of impact boundaries, *Discontinuity Boundaries* with dynamics represented by convex combinations of the solution of the ODEs of each vector field and *Impact boundaries* with dynamic represented by algebraic equations. Eq.(1) shows the state-space representation of the simplest nonsmooth system with the three types of dynamics.

$$\dot{\mathbf{x}} = \left\{ \begin{array}{l} \mathbf{F}_i(\mathbf{x}, \alpha) \\ if \quad \mathbf{x} \in Z_i = \{\mathbf{x} \in \mathbb{R}^n : H(\mathbf{x}, \alpha) > 0\} \\ \mathbf{F}_j(\mathbf{x}, \alpha) \\ if \quad \mathbf{x} \in Z_j = \{\mathbf{x} \in \mathbb{R}^n : H(\mathbf{x}, \alpha) < 0\} \\ \mathbf{G}(\mathbf{x}, \alpha) \\ if \quad \mathbf{x} \in \sum_{i,j} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : H(\mathbf{x}, \alpha) = 0\right\} \\ I(\mathbf{x}, \alpha) \\ if \quad \mathbf{x} \in \sum_{(i,j,k)} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : H_I(\mathbf{x}, \alpha) = 0\right\} \end{array} \right\} \tag{1}$$

In Eq.(1) $\mathbf{F}_i$ and $\mathbf{F}_j$ are smooth vector fields; Z_i and Z_j are the corresponding regions and $\alpha \in \mathbb{R}^1$ is a parameter. The state space regions are determined by the smooth scalar function $H(\mathbf{x}, \alpha)$ and the boundary of impact of Z_i or Z_j regions is determined by the scalar function $H_I(\mathbf{x}, \alpha)$.

3.1 Zero degree of smoothness systems

In electro-mechanical nonsmooth systems the impact phenomena is highly dynamical, then can be declared using an algebraic relation due to the impact time is negligible in relation with the time constant of mechanical systems. In this relation, γ is the restitution coefficient and $\dot{\Omega}_I^{(-)}$, $\dot{\Omega}_I^{(+)}$ are respectively the approximation and bounce speed.

$$I(\mathbf{x}, \alpha) = \begin{cases} \Omega_I^{(+)} = \Omega_I^{(-)} \\ \dot{\Omega}_I^{(+)} = \gamma \dot{\Omega}_I^{(-)} \end{cases} \tag{2}$$

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The first row of Eq. 2 expresses that the position before and after the impact are identical. The second one expresses that the rebound velocity (+) equals the impact velocity (-) multiplieed by the restitution coefficient γ.

3.2 First degree of smoothness systems

Filippov systems, a set of first-order ordinary differential equations with a discontinuous right-hand side are a subclass of discontinuous dynamical systems. The trajectory of a sliding orbit remaining partially inside the discontinuity boundary may be calculated by the Filippov convex method as in [4]. Systems with multiple regions and *DBs* are treated in [13], where an extended equation for Filippov systems is described in order to deal with the intersection of several discontinuity surfaces.

In Filippov systems, between Z_i and Z_j in the discontinuity boundary, we assume that there is a region $\sum_{i,j}$, which are a vector field of $\mathbb{R}^{n-1}$ dimension conformed by three types of points: crossing (Ω_C), sliding (Ω_S) and singular (Ω_{SO}), and each one with subtypes. The scalar function $\sigma(\mathbf{x})$ is used to determine the point type, according to the geometric condition of the vectors in the $\mathbf{x}$ point of analysis. Eq. 3 describes the geometric conditions of an sliding point. Eq. 4 helps to determine which is the nature of the point, according to the value of $\sigma(\mathbf{x})$ and the neighboring vector fields at $\mathbf{x}$.

$$\sigma(\mathbf{x}) = \{\langle H_x(\mathbf{x}), \mathbf{F}_i(\mathbf{x}, \alpha)\rangle \langle H_x(\mathbf{x}), \mathbf{F}_j(\mathbf{x}, \alpha)\rangle\} \tag{3}$$

$$\mathbf{x} \in \Sigma_{i,j} : \begin{cases} \Omega_C \Rightarrow \sigma(\mathbf{x}) > 0 \\ \\ \Omega_{SO} \Rightarrow \\ (\sigma(\mathbf{x}) = 0) \wedge (\langle H_x(\mathbf{x}), \mathbf{F}_j(\mathbf{x}) - \mathbf{F}_i(\mathbf{x})\rangle = 0) \\ \\ \Omega_S \Rightarrow \sigma(\mathbf{x}) < 0 \\ \end{cases} \tag{4}$$

Crossing points (Ω_C), characterized by $\sigma(\mathbf{x}) > 0$, are points which the evolution of the trajectory will not remain in the *DB*. Instead, it crosses from the region in which has been previously evolving to the other.

Singular sliding points (Ω_{SO}), characterized by $\sigma(\mathbf{x}) = 0$, are points having the associated vectors with the normal component $\langle H_x(\mathbf{x}), \mathbf{F}_i\rangle$ equal to $\mathbf{0}$. This is because the vectors are tangential to the *DB* or vanishes. At such points: (a) $\mathbf{F}_i$ and $\mathbf{F}_j$ are tangent to the *DB*, (b) either $\mathbf{F}_i$ or $\mathbf{F}_j$ vanishes while the other is tangent to the *DB*, or (c) $\mathbf{F}_i$ and $\mathbf{F}_j$ vanish. To avoid the lack of definition of the Filippov solution for these points, in the examples, we adopt the methods presented in [14] which coincide with the topology of the normal forms VV, VI and II presented in [12].

Sliding points (Ω_S) are characterized by $\sigma(\mathbf{x}) < 0$. When a sliding motion is presented in the discontinuity boundary, the Filippov method gives as a solution a tangent vector to the *DB* which is a convex combination $\mathbf{G}(\mathbf{x}, \alpha)$, of the vector fields $\mathbf{F}_i$ and $\mathbf{F}_j$ at a point $\mathbf{x} \in \Sigma_{i,j}$ (Eq. 5).

$$\mathbf{G}(\mathbf{x}, \alpha) = \lambda \mathbf{F}_i(\mathbf{x}, \alpha) + (1 - \lambda)\mathbf{F}_j(\mathbf{x}, \alpha) \tag{5}$$

$$\lambda = \frac{\langle H_x(\mathbf{x}), \mathbf{F}_j(\mathbf{x}, \alpha)\rangle}{\langle H_x(\mathbf{x}), \mathbf{F}_j(\mathbf{x}, \alpha) - \mathbf{F}_i(\mathbf{x}, \alpha)\rangle} \tag{6}$$

λ is a scalar function defined through the projections of the vector fields in the direction of the normal vector ($H_x(\mathbf{x})$) to the discontinuity boundary. According to the direction of the normal components of the vectors, the sliding points are stable (or attractor) (Ω_{SS}), or unstable (or repulsive) (Ω_{SU})(Eq. 7).

$$\mathbf{x} \in \Sigma_{i,j} : \begin{cases} \Omega_{SS} \Rightarrow \\ (\langle H_x(\mathbf{x}), \mathbf{F}_i\rangle > 0) \wedge (\langle H_x(\mathbf{x}), \mathbf{F}_j\rangle < 0) \\ \Omega_{SU} \Rightarrow \\ (\langle H_x(\mathbf{x}), \mathbf{F}_i\rangle < 0) \wedge (\langle H_x(\mathbf{x}), \mathbf{F}_j\rangle > 0) \end{cases} \tag{7}$$

From Eq.(4) the crossing set is open but the sliding set is closed, it is the union of the sliding segments, singular points and isolated or special sliding points. In this paper, the terms *special points* or *isolated points* refer to points whose neighbor points belong to a different class.

Special points define important dynamics in the sliding segments of 2d systems or areas in 3D systems. These points are: (a) Equilibria points, in which both vectors $\mathbf{F}_i$ and $\mathbf{F}_j$ are attractive, transversal to the *DB* and are at the end of two sliding segments pointing each other. (b) Quasi-equilibria points with both vectors $\mathbf{F}_i$ and $\mathbf{F}_j$ attractive transversal or anti collinear and which are at the start of two sliding segments pointing away each other. The contrary case have also quasi equilibria points: repulsive, transversal points which are at the end of two sliding segments pointing each other. (c) Boundary equilibria points, in which one of the vector $\mathbf{F}_i$ or $\mathbf{F}_j$ vanishes. (d) Tangent points, in which one of the vectors $\mathbf{F}_i$ or $\mathbf{F}_j$ is tangent to the *DB*. In [15] is done a more strict classification giving the characterization of 42 types of points with the objective of differentiate topologies in order to detect bifurcations.

3.3 Second degree of smoothness systems

The second degree of smoothness systems are represented as variable structure systems having different dynamics in each zone or region. The dynamics of the system does not allow sliding or stops on the boundary zone, all points are crossing and hence, there is not a particular dynamics defined in the limit zone, instead there is a change of the region equations set.

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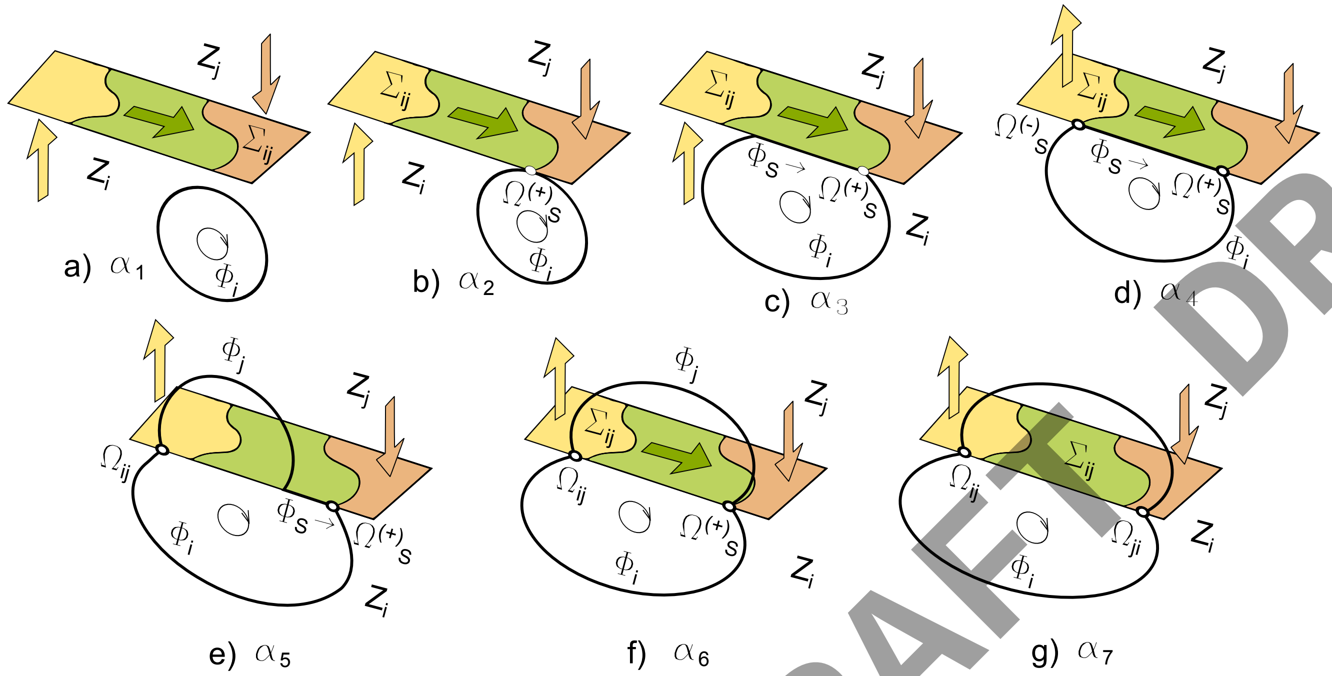

Figure 1: Grazing (a,b,c), Switching (c,d,e) and Crossing (e,f,g) bifurcations.

4 Sequences of well known bifurcations

In this and the following sections, we will present the cycles of the most referenced sliding bifurcations as sequences of elements. In each cycle are presented the constituent elements assuming that its presence was detected, in the same order, in the evolution of a dynamical system. In next equations the symbol Φ is used to represent segments composed by the same type of point. Arrows indicate the direction of the sliding segments related to the *DB*.

4.1 Grazing Bifurcation

The Grazing Bifurcation (${}^s\beta_G$) occurs in the following sequence of changes. First, there is an orbit of a limit cycle C_1 evolving in only one of the regions i or j, without hitting the boundary, as shown in Fig. 1(a).

$$C_1 = \Phi_i / \circlearrowright \tag{8}$$

Then, when the parameter α changes, for example, from α_1 to α_2, the cycle grows or moves toward the discontinuity and has a tangent contact with the last point of a sliding segment $\Omega_s^{(+)}$. The structure presented corresponds to a ${}^sC_2$ type cycle.

$${}^sC_2 = \Phi_i / \Omega_s^{(+)} / \circlearrowright \tag{9}$$

Subsequently, as the parameter is moved further, the limit cycle changes again as is depicted in Fig. 1(c). The structure presented corresponds to a ${}^sC_3$ type cycle.

$${}^sC_3 = \Phi_i / \Phi_s^{\rightarrow} / \Omega_s^{(+)} / \circlearrowright \tag{10}$$

The orbit of the limit cycle ${}^sC_3$ has now two different pieces: one without touching the discontinuity boundary and the other one, corresponding to a sliding segment $\Phi_s^{\rightarrow}$ that starts in any intermediate point of the discontinuity and ends at a tangent point $\Omega_s^{(+)}$. The equation describing the sequence of cycles is:

$${}^s\beta_G = (C_1)\,[{}^sC_2]\,({}^sC_3) \tag{11}$$

Impact systems also present grazing bifurcations. An orbit that is evolving in a region, due to a change in a parameter, makes contact with a boundary in only one point. This point has approximation speed equal to zero. Consequently, the bouncing speed is also zero. If the physical parameter continues changing, the approximation and rebound points separate. The corresponding cycles are:

$$\begin{aligned} C_1 &= \Phi_i / \circlearrowright \\ {}^iC_2 &= \Phi_i / \Omega_I^{(+-)} / \circlearrowright \\ {}^iC_3 &= \Phi_i / \Omega_I^{(+)} / \Omega_I^{(-)} \circlearrowright \end{aligned} \tag{12}$$

4.2 Switching Bifurcation

The sequence of changes for a Switching Bifurcation (${}^s\beta_S$) is as follows: the sliding piece of a limit cycle of type ${}^SC_3$ grows until it reaches the first point $\Omega_S^{(-)}$ of the sliding segment. See Fig. 1(d). The type of structure presented, corresponds to a cycle ${}^sC_4$. In general,

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the second cycle always characterizes the bifurcation type and it is only presented for one value of the parameter or a very narrow range in the numerical calculation terms.

$$^{s}C_4 = \Phi_i/\Omega_s^{(-)}/\Phi_s^{\rightarrow}/\Omega_s^{(+)} \circlearrowright \quad (13)$$

With a further change in the parameter the orbit has now three segments: two of them, Φ_i and Φ_j are in two different regions separated by the discontinuity boundary, and the third piece is on the sliding region moving to the right. See Fig. 1(e). The structure presented corresponds to a $^{s}C_5$ type cycle.

$$^{s}C_5 = \Phi_i/\Omega_{C_{i,j}}/\Phi_j/\Phi_s^{\rightarrow}/\Omega_s^{(+)}/ \circlearrowright \quad (14)$$

The equation describing the sequence of cycles is:

$$^{s}\beta_S = (^{s}C_3)\,[^{s}C_4]\,(^{s}C_5) \quad (15)$$

4.3 Crossing Bifurcation

A Crossing Bifurcation ($^{s}\beta_C$) occurs when the sliding piece of a cycle $^{S}C_5$ gets smaller and smaller. At a parameter value α_6, the piece of trajectory Φ_j hits the sliding region just at the last point of the sliding segment $\Omega_s^{(+)}$. See Fig. 1(f). The structure presented corresponds to a $^{s}C_6$ type cycle.

$$^{s}C_6 = \Phi_i/\Omega_{C_{i,j}}/\Phi_j/\Omega_s^{(+)}/ \circlearrowright \quad (16)$$

As the parameter further changes at some value α_7, the limit cycle has now two pieces without sliding. The structure presented corresponds to a $^{s}C_7$ type cycle.

$$C_7 = \Phi_i/\Omega_{C_{i,j}}/\Phi_j/\Omega_{C_{j,i}}/ \circlearrowright \quad (17)$$

The equation describing the sequence of cycles is:

$$^{s}\beta_C = (^{s}C_5)\,[^{s}C_6]\,(^{s}C_7) \quad (18)$$

4.4 Adding or Multisliding bifurcation

The sequence of changes for the Adding or Multisliding bifurcation is related to the addition or destruction of a second sliding segment in the discontinuity boundary as is described in [12]. Other sliding bifurcations recently reported are those including more than two discontinuity boundaries that are moving due to variations of a parameter. Those ones were introduced in [8] using an example.

5 The implementation of the sequences as a method of comparison

Next we will describe the tool which were developed to get the results obtained in the previous section. Additional to the numerical integrator, there are some databases, procedures and methods running in parallel. They perform the evaluation of information collected previously, and the information acquired in real time, when the system is evolving. These tools are:

5.1 Collection of points

The collection of the values of the points is done in a vector, called `vector_of_states`. The new point includes the values of the states, the amount of time since the integration started and the data of the vector fields involved in the dynamics. As shown in Fig.3(a), after each iteration of the numerical integration, one point is added to the vector of states and the graphic of the space states.

5.2 Database of point characteristics

Each point, additional to the characterization given by the states is classified by the region or DB it belongs. The orientation of the two vector fields for points in the DB determines types as anticollinear, transversal, tangent, also the attractiveness or repulsiveness and the direction relative to the DB. The magnitude of the vectors might tend to zero. The equation 4 determines if is a crossing or sliding point. Finally the equation 1, that represents its dynamics indicates if is an impact point. All points and their characteristics are listed in a 2x2 array called `matrix_of_points`, where the first column is the list of points and each row are the list of attributes that each point should to fulfill [15]. Other points presenting themselves in the evolution belonging only to one region, are the nodes and focus, stable and unstable.

5.3 Recognition of Points

From the states of the points and vector fields involved, secondary information is estimated. For a point in the DB is evaluated if it is impacting or normal. Then is evaluated if the point is crossing or sliding. If a point is crossing, it is evaluated to which vector field the evolution will move. The evolution of sliding points has direction tangent to the DB, spanning 42 possible subtypes [15]. Summarizing, each point should match all attributes listed in a row of the point matrix. The detected points are stored in `vector_of_elements` (Fig. 2).

- While the vector of elements is being filled out other functions are debugging the information. Each point in a cell of the vector of elements is compared with

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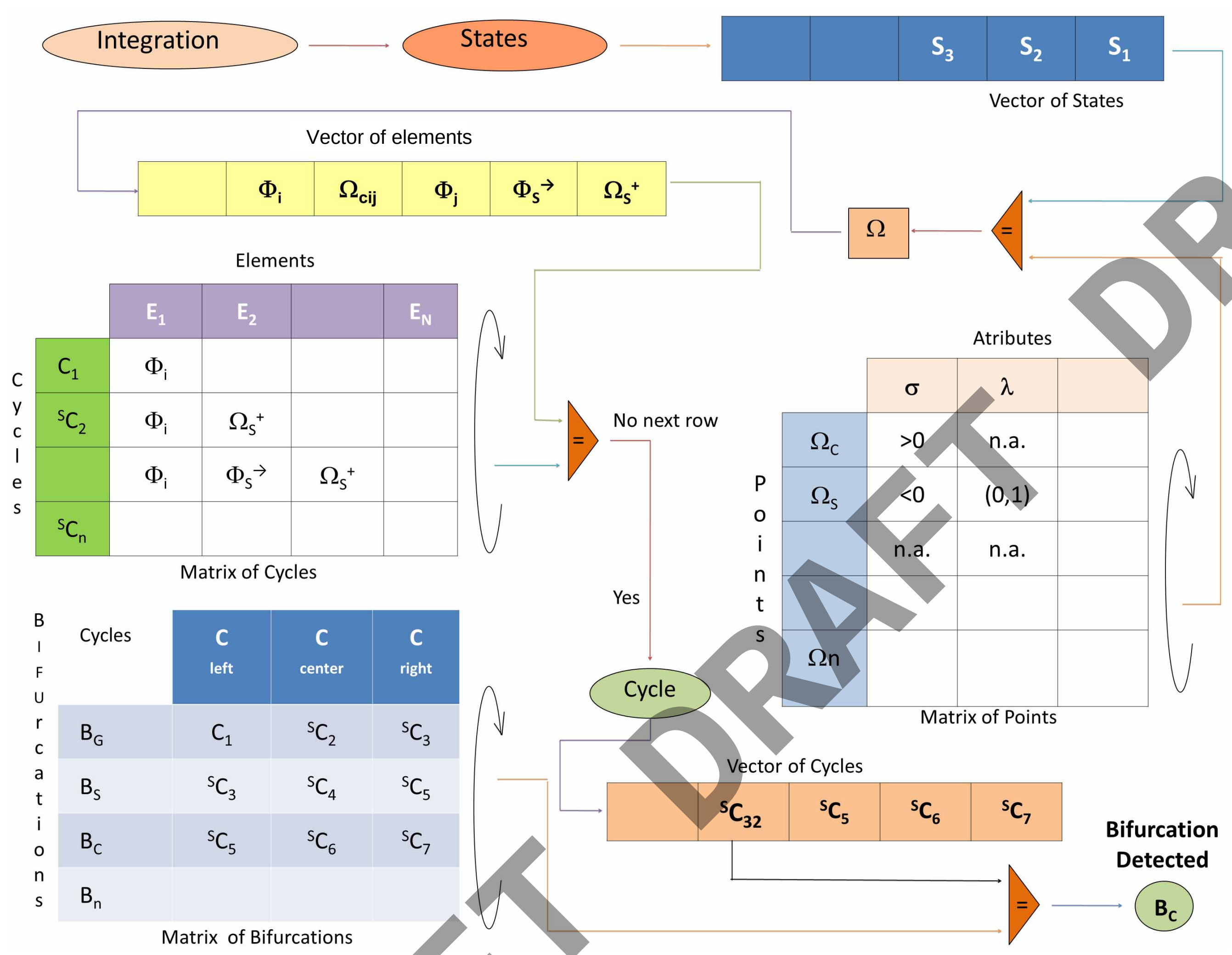

Figure 2: General method of comparison of sequences of cycles and bifurcations.

the point that was met immediately before. Data of points having equal identity are removed from the vector. Instead, the repetition of points turns the first point in the repetition into a piece of curve of the same type. This procedure is carried out with the objective of avoiding a situation in which the vector is filled or saturated with the same data.

- While picking elements for the matrix, events with wrong result can be found and should be corrected. For example, it is impossible to accept the sequence Φ_i/Φ_j because implies a change of region Z_i to Z_j. In the change, a crossing point must be found, and an admissible sequence would be $\Phi_i/\Omega ij/\Phi_j$. Thus, a function to correct the sequences of elements is necessary. In [16] are listed 51 rules to correct errors.

5.4 Database of cycle elements

Each cycle as presented in the previous section, has a set of elements which could be points or segments of points. The order of the elements also determines the cycle. In order to have a wider data base all papers in the literature should be analyzed and the cycles presented must be converted in sequences of elements. The information is stored in a bidimensional array, called `matrix_of_cycles`, in which each row are the identities of the elements of a cycle.

5.5 Comparison of cycles

In this step the comparison between the matrix of cycles and the vector of elements is performed. We wish to know whether inside the vector of elements there exists a sub-vector of consecutive and ordered elements that matches with some row of the matrix of cycles. The sequence in the appearance of cycles (and other dynamics) in this step is recorded in a vector called `vector_of_cycles`. The result in the vector of cycles, for a given set of parameters, admits the presence of (a) equilibrium points, (b) limit cycles, and (c) chaotic behavior. For time-varying parameters, the system evolution might be

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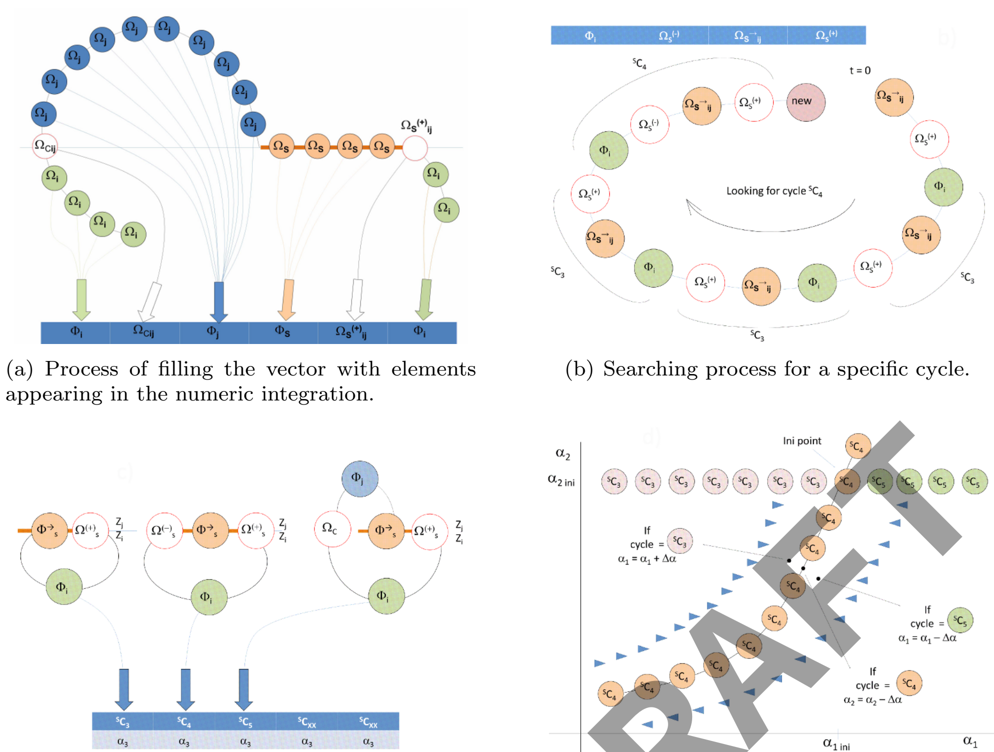

(a) Process of filling the vector with elements appearing in the numeric integration.

(b) Searching process for a specific cycle.

(c) Cycle tracking process for bifurcations detection.

(d) Cycle continuation process.

Figure 3: Implementation of Cycle Bifurcation.

a sequence of n cycle types, whose order is dictated by the system nature (Figs. 3(b) and 3(c)).

To prevent that a repetition of a cycle be mistaken as a single cycle, a function running in parallel with the integrator performs the evaluation and the correction. When a sub-sequence of the vector of elements, beginning in the position nb_1, is equal to the sub-sequence beginning in the position $nb_2 = nb_1 + l_j$ and l_j is the number of elements of the cycle, it is concluded that a cycle is repeating. A cycle is completed when a sequence of elements is continuously repeated and the time Γ to repeat becomes constant. Let us assume, as illustration, a sequence with a grazing cycle $\Phi_i/\Omega_s^{(+)}$. After some time 3Γ, the matrix of elements would contain a cycle with the sequence $\Phi_i/\Omega_s^{(+)}/\Phi_i/\Omega_s^{(+)}/\Phi_i/\Omega_s^{(+)}$, which is not correct.

If the search is for a specific cycle, the procedure is slightly different. In this case, the number of elements in the cycle under consideration is a date and then it is reserved the same amount of cells to store the elements during the integration process. When a new element appears, a comparison is carried out until all the elements of the stored cycle are identical to the elements that are picked up from the integration (Fig.3(b)).

5.6 Change in parameter and storing of cycles

When a cycle is already stored in `vector_ of_cycles` and it is continuously repeating, a programmed disturbance is introduced in a physical parameter, to continue searching the bifurcations. The previous processes are repeated, and recorded in `vector_ of_cycles`.

5.7 Database of cycles sequence

Each bifurcation is constituted by three ordered cycles, the first and third are presented for a wide range of the parameter but the second is only presented for a value of the parameter. The information of the bifurcations is then stored in a bidimensional array, called `matrix_of_bifurcations`, in which each row are the identities of the three cycles of the bifurcation.

5.8 Comparison of cycles sequence

The objective of the comparison is to identify if inside the vector of cycles there is a sub-vector of three consecutive and ordered cycles which matches a row of the bifurcation matrix (Fig. 3(c)). Here we are looking for a specific sequence that corresponds to a known bifurcation. To achieve this, a double comparison must be performed: the first part is the comparison of elements that

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forms cycles, and the other part is referred to the comparison of the behavior of cycles in a specific sequence, until a full match is detected. When the phenomenon is poorly understodd, the comparison could be used to identify sequences of cycles which occur when a parameter is modified within a range. For this purpose, the integrator uses the vector of cycles to store information regarding the cycles which have been found during the time that the method has been active. Each time the integrator detects a repeated sequence of elements, stores the information of the cycle, and changes the parameter value in order to continue with the next identification.

5.9 Continuation

To continue a bifurcation the parameters are adjusted corresponding to the central cycle of a previously detected bifurcation. Next, two additional parameters are slightly changed as per the rules of continuation. The first parameter is disturbed and the second changes accordingly, to keep the dynamics of the central cycle. This controlled disturbance of the two parameters is repeated, such that it determines a trajectory in a continuation-plot. The change of parameters could be done using methods like predictor-corrector described in [17] or [18]. In this cases the predictive function is the cycle that generates the bifurcation, and the previous and posterior cycles to the bifurcation are used for correction.

Fig. 3(d) shows an example of how is used the method of comparison. The first step is a sensibility analysis that indicates to which cycle, the system evolves when the parameters are increased or decreased. For example, the bifurcation ${}^{S}C_2$ has a sequence of cycles $\left({}^{S}C_3\right)\left[{}^{S}C_4\right]\left({}^{S}C_5\right)$. Assume that a direct proportional sensibility exists for parameter α_1. This implies that a small increment in the parameter value tends to change the cycle into ${}^{S}C_5$ and a small decrement tends to change the cycle into ${}^{S}C_3$. Changing α_1, the cycle ${}^{S}C_4$ is obtained. Then, the second parameter α_2 is decreased (in this case the initial point has a high value). After the change in parameter α_2, the cycle ${}^{S}C_4$ changes to ${}^{S}C_3$ or to ${}^{S}C_5$. In the first case, the continuation algorithm increases α_1 until the cycle type ${}^{S}C_4$ is found again. In the second case, the algorithm acts conversely. The process is continuously iterated until the prescribed final value of parameter α_2 is reached.

Two objectives of an application for automatic bifurcation detection are: (1) to perform the detection task without a close supervision, and (2) to track bifurcations through continuation. The procedures developed here can be used to achieve these goals.

6 Conclusion

This article presents an alternative method for detecting bifurcations of limit cycles in non-smooth systems. We focused on complex systems, which defy boundary-value methods. The comparison method, reported in this article, is not intended to focus in the same achievements of other methods. Instead, it addresses open issues left by them, such as multiple sliding segments and discontinuity boundaries (DB). The comparison method differs from other approaches in the identification and manipulation of the system information. While the methods in [10] and [11] consider a system as one entity to be solved by a group of equations, the comparison method uses previously collected information in a data base of points, cycles and bifurcations. This information allows comparisons and decision making. To enable the method for nonsmooth systems, the cases when the evolution crosses the *DBs* of systems having simultaneously the three degrees of smoothness (impact, Filippov and first derivative discontinuities) was analyzed. To achieve the goal was used the method that characterizes and records the elements comprising the cycles in the order they appear in the integration process. The cycles were characterized as sequences of elements (points and segments). It must be noticed that the sequence of cycles has the topological changes (e.g. bifurcations) implicit. Some of the types of data considered as topological characteristic and collected during the evolution are: (a) number of elements of the cycle, (b) order in which the cycle elements are generated, (c) position of the sliding elements in the sequence of cycle generation, (d) way (e.g. extreme or interior) in which the cycle reaches and leaves the sliding segment, (e) discontinuity boundary to which the element belongs, (f) direction (CW, CCW) in which the cycle evolves. In this article we also report a textual notation to describe the elements of the cycles. The comparison method is also able to handle continuation of sliding bifurcations.

The method of comparison could be implemented using tools of the sequence theory, suffix-trees and string-matching, which offer procedures to drive a large number of elements and allow us to discriminate subsets with low computing time investment. The procedure of comparison fulfill the two tasks required by an application for automatic bifurcations detection: perform the detection task without a closed supervision and track bifurcations through continuation.

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