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Graph-Based Structural Analysis of Planar Mechanisms

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Abstract Structural analysis of planar mechanisms deals with the enumeration of distinct mechanisms derived from a kinematic chain and their mobility characterization. The problem of finding the *generation principle of a mechanism* is central to the structural analysis and consists of determining a sequence of kinematically and statically independent-simpler modules, this sequence represents the mechanism's topology and provides a transferring direction of information for kinematics and force analyses. This article presents a structural analysis of planar mechanisms using graph theory followed by a novel graph-based algorithm to determine the generation principle of planar mechanisms with closed-loop kinematic structure. The running time analysis and proof of correctness of the algorithm are provided and its validation is carried out using a case study.

Keywords graph · kinematic structure · Assur group

1 Introduction

Modular kinematics and force analyses of mechanisms are reported to be computationally efficient and adaptable since they allow to define in advance a library of modules that are combined and reused to model wide families of mechanisms [8, 14]. Although modular analyses methods have been available in the literature for several decades, interest on these kinematic [5–7, 9–11, 20, 21] and dynamic analyses methods [4, 8, 18, 19] remains current. Recently, modular kinematic analysis has been extended to calculation of Jacobian matrices and quality indexes [2, 3]. Similar to the analysis, synthesis of mechanisms that deals with topological variations often implements modular approaches [6, 13, 14] that allow to rebuild instantly the kinematic structure of the model, therefore, ensuring the continuity of the design process [19]. In this sense, modules (*e.g.* Assur groups) are building blocks or structural genes of complex systems (*e.g.* mechanisms) [15] used, for example, for the modeling of biological systems and the designing of prosthetic devices [16].

The problem of finding the *generation principle of the mechanism* or *system group classification* is central to modular analysis and synthesis methods [14, 17, 19] and consists of the determination of the modules that form a planar mechanism and the sequence in which those modules are connected. Only two categories of modules are considered for this work:

Definition 1. *Driving-elements group* which is formed by the fixed link and the input links. It represents the generalized coordinates of the system.

Definition 2. *Assur group* which is a planar kinematic chain composed of $n = 2k$ links and $j = 3k$ joints and with the following characteristics:

a. each joint has one mobility,
b. there are r external joints that connect the group to an external kinematic chain with mobility M,
c. the mobility of the expanded kinematic chain (external kinematic chain plus Assur group) remains M, and
d. it is not possible to decompose the kinematic chain into simple chains so that all fulfill the same requirements.

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The generation principle is determined by the topology of the mechanism and its input-motion scheme. From that point of view, this problem can be consider as laying in the kinematic-structure problem category, particularly, in the structural analysis of mechanisms.

Traditionally, the generation principle of the mechanism is determined by inspection. However, a systematic approach that allows for computer implementation is required when complex mechanisms are treated. Alternative strategies aimed to perform this task are developed in [17] and [1]. In spite of the fact that both alternatives are based on a graph representation of the mechanism, the functions applied to map mechanisms to the graphs are quite different. The mapping function implemented in [1] has a kinematical background, whereas the function used in [17] is based on rigidity theory.

This article presents a graph-based formulation of the generation principle of mechanisms. The formulation has a kinematical background and it includes formal definitions of the graphs representing an Assur group and a driving-elements group (section 3.2). In addition, these definitions are implemented in section 3.4 into the design of a novel algorithm to determine the generation principle of planar mechanisms. The proof of correctness of the algorithm and its running time analysis are provided in section 3.5.

2 Literature Review

Main aspects that encompass the kinematic-structure analysis of mechanisms include mobility analysis and Degrees of Freedom (DOF), structural synthesis of kinematic chains, isomorphism of kinematic chains, structural analysis, and application to the creative design of products and systems [12].

Mruthyunjaya [12] presents a comprehensive study of available literature concerning the research on kinematic structure of mechanisms in which the aforementioned analyses and syntheses aspects are covered. Particularly, structural analysis deals with the enumeration of distinct mechanisms derived from a kinematic chain and the characterization of their freedom type depending on the input and output motion scheme.

A different category of structural analysis deals with the concept of generation principle of mechanisms which is derived from the classical concept of kinematic structure developed by Leonid Assur in 1914. An Assur group is a minimal kinematic chain with zero mobility from which it is not possible to obtain a simpler kinematic chain with the same mobility, see definition section 1 definition 2. Usually, a mechanism can be designed as the successive joining of a driving-elements group (formed by the fixed and input links) and several Assur groups. The Assur groups that form a mechanism mechanism and their order of succession determine the kinematic structure of the mechanism. Assur groups are statically determined. Thus, they represent a modular basis for the kinematics and force analysis, and for the synthesis of mechanisms [18].

Although research on kinematic structure of mechanisms remains of interest, there is few literature available on strategies (algorithms) for the determination of the generation principle of mechanisms based on Assur-group decomposition.

Buśkiewics [1] presents a graph-based combinatorial algorithm in which the Assur groups forming a planar mechanism are systematically identified. The algorithm precondition establishes that the internal loops' combinatorial of a graph that represents a mechanism includes its Assur groups. It is computationally efficient since practical mechanisms conduct to a small combinatorial of internal loops. However, it works under the condition that the set of independent loops of the representing graph is given, which implies a combinatorial task itself.

Offer Shai and coworkers [17] develop a study in which Assur groups are characterized as graphs having special properties from rigidity theory. The article establishes the duality between planar (locked) mechanisms and unstable isostatic frameworks, leading to the implementation of rigidity-theory mathematical tools into the kinematics and force analysis of mechanisms. Moreover, a combinatorial algorithm for determination of the generation principle of a mechanism is developed. For this purpose, a rather unusual function for mapping mechanisms to graphs is implemented, in which the kinematic pairs are represented by vertices and the links by edges. Although, the function allows for a direct use of the rigidity theory in the graph-based analysis, a conflict with a kinematics fundamental arises for the representation of multiple-joint links: only binary links are considered in the representation, therefore, multiple-joint links are represented as a number of different binary rigid bodies connected between them, even though no relative motion is developed.

This work presents an alternative to the determination of the generation principle of mechanisms. A novel graph-based algorithm is designed in which mechanisms are represented by graphs in a conventional manner, resulting in the direct mapping of the kinematic-structural properties to the graph representation. For this purpose, kinematic properties of planar mechanisms and Assur groups are rigorously defined in both mechanisms and graphs domain. Section 3 develops the mapping from mechanisms to graphs, the fundamentals of the kinematic-structure analysis, and the algorithm to determine the generation principle of mechanisms.

3 Methodology

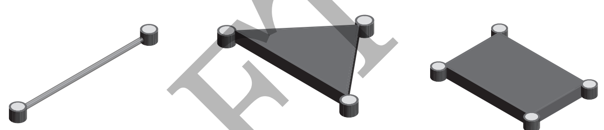
3.1 Mapping mechanisms to graphs

The kinematic structure of planar mechanisms can be represented in several different ways. In this section, two principal representation methods (structural and graph representation) are described. For the sake of simplicity, the following assumptions are made for both representation methods:

a. Mechanisms with excessive constraints will be represented without their excessive constraints.
b. All joints are assumed to be binary. Multiple joints will be replaced by a set of equivalent binary joints.
c. Two mechanical elements with no relative motion between them will be represented as one link.
d. All joints will be assumed to have only one mobility.

3.1.1 Structural representation of mechanisms

In a structural representation, each link of a mechanism will be illustrated by a polygon whose vertices represent the kinematic pairs. In this manner, a *binary* link is represented by a line with two end vertices standing for their kinematic pairs, a *ternary* link is represented by a triangle with three vertices, a *quaternary* link is represented by a quadrilateral with four vertices (see Fig. 1), and so on. Furthermore, in structural representation of a mechanism, the links are numbered, particularly, the fixed link is numbered with zero.



(a) Binary link (b) Ternary link (c) Quaternary link

Fig. 1 Structural representation of links

Figure 2 presents the structural representation of a four bar mechanism.

3.1.2 Mapping function

A mechanism is a set of links connected by a set of joints. Therefore, it can be conveniently represented as a graph. This usually simplifies the structural analysis since many graph properties can be applied directly and

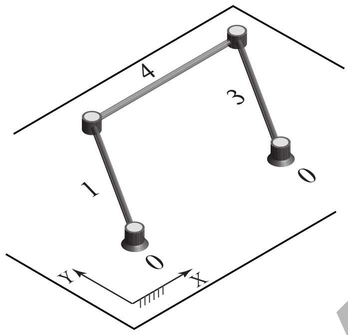


Fig. 2 Structural representation of a four-bar mechanism

graph theory may be used as an aid for computer implementation. For this purpose, we define the mapping function g in (1) such that it takes a kinematic chain K and returns a graph G in the following manner:

$$\begin{aligned} g :& K \longrightarrow G, \\ g(N, J) =& (V, E) = (V(N), E(J)), \end{aligned} \tag{1}$$

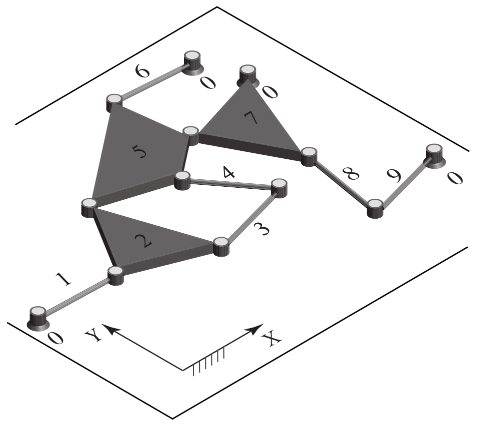
where:

N is the set of links of the mechanism,
J is the set of joints of the mechanism,
V is the set of vertices of the graph, and
E is the set of edges of the graph.

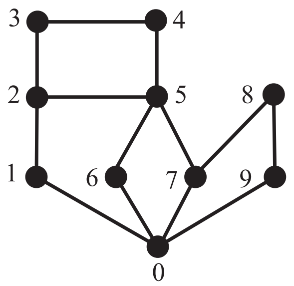
In graph representation, the vertices of the graph denote the links of the mechanism and the edges denote the joints. Particularly, a joint connecting links u and v is represented by the edge $e = (u, v)$. In such a case, u and v are called the end points of e. The correspondence between the graph vertices and their respective links is straight forward.

3.1.3 Graph representation of mechanisms

The edge connecting two vertices in graph representation corresponds to the pair that connects two links. The degree of a vertex (number of edges incident with that vertex) represents the classification of its corresponding link. In this manner, we call a vertex of degree two a *binary* vertex (representing a *binary* link), a vertex of degree three a *ternary* vertex, and so on. We also call the graph representing a mechanism $G = (V, E)$. The joints connecting the driving links with the fixed link or other driving-elements are represented by thicker edges. Figure 3 shows a mechanism (Fig. 3.a) and its graph representation (Fig. 3.b).



(a) Mechanism consisting of 9 links and 13 joints



(b) Graph representation of Fig. 3(a)

Fig. 3 Mechanism consisting of 9 links and 13 joints together with its graph representation

3.2 Structural analysis of planar mechanism

3.2.1 Domain and range of mapping

The herein developed structural analysis concerns those mechanisms that have a closed-loop kinematic structure and their input links connected either to the fixed or to another input link. According to this assumption, a graph obtained after mapping such type of mechanism, by the use of (1), presents the following characteristics:

a. It is a connected graph. This means that every vertex in the graph is connected to every other vertex by at least one path.
b. It is a block. This is, the graph is connected and has no cut points or bridges. A cut point is a vertex whose removal results in an increase in the number of components. In addition, a bridge is an edge whose removal results in an increase in the number of components. Bridges and edges often arise when representing a mechanism with excessive constraints in graph form.
c. It has no parallel edges or slings as only binary joints are accounted for and two links with no relative motion between them are represented as one link.

3.2.2 Degrees of freedom and independent loops of a mechanism

The first concern in the study of a mechanism's kinematic structure is the DOF. For a graph $G = (E, V)$ which represents a mechanism M that meets conditions stated in section 3.1.2, the DOF W of a mechanism can be written as (2):

$$W = 3(|V| - 1) - 2\,|E|\,, \tag{2}$$

where:

W is the DOF of the mechanism,
$|V|$ is the number of vertices, equivalent to the number of links of the mechanism, and
$|E|$ is the number of edges, equivalent to the kinematic pairs of the mechanism.

The term $(|V| - 1)$ results from taking vertex zero out the DOF count. In addition, it is possible to determine the number of independent loops L in the graph of a mechanism by considering the equation of Euler (3):

$$L = |E| - |V| + 1. \tag{3}$$

3.2.3 Structure and generation principle of planar mechanisms

Several methods focused on the analysis of planar mechanisms rely on the study of their kinematic structure, particularly, on their division into simple parts or groups of elements called *driving-elements group* and *Assur groups*.

The DOF of a mechanism can be written as:

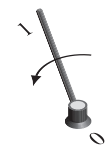
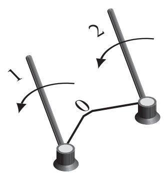
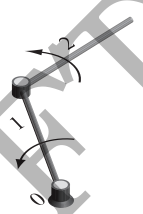
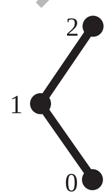
$$W = W + 0 + 0 + \ldots + 0. \tag{4}$$

According to (4), a mechanism can be divided into separate parts or kinematic chains. The simpler kinematic chain, whose DOF W is equal to the DOF of the whole mechanism is denoted as the driving-elements group. Those kinematic chains whose DOF is equal to zero and which cannot be disaggregated into simpler kinematic chains with the same property are denoted as Assur groups, see section 1, definitions 1 and 2. In this sense, a planar mechanism can be considered as consisting of a driving-elements group and a number, one or more, of Assur groups. We call a graph representing an Assur group $G_A = (V_A, E_A)$, where $G_A \subset G$. The Assur groups forming a mechanism and the order in which those groups are connected between them determine the generation principle.

3.2.4 Driving-elements group

The driving-elements group consists of one fixed element and one or more free elements called *primary elements*. The primary elements are connected either with the fixed element or with another primary element and the DOF W of the mechanism is equal to its number of primary elements. The graph that represents a driving-elements group is called $G_D = (V_D, E_D)$, where $G_D \subset G$, and its characteristic is that every vertex is connected with vertex zero by one path. Table 1 shows different types of driving-elements groups consisting of one and two primary elements together with their graph representation.

Table 1 Primary-elements groups and their graph representation

Primary-elements number	Structural representation	Graph representation
1		
2		
2		

3.2.5 Assur groups

The graph representing an Assur group $G_A = (V_A, E_A)$ is a connected graph and its DOF can be determined based on (2). For this purpose, it is necessary to consider that every element of the group is movable as a result of incorporation of the fixed element into the driving-elements group. The relation between the edges and vertices of a graph representing an Assur group is given by (5) since the DOF of the Assur group is zero with respect to the links that join to this group:

$$3\,|V_A| - 2\,|E_A| = 0. \tag{5}$$

Both numbers $|V_A|$ and $|E_A|$ must be natural, thus, the product $2\,|E_A|$ will always be an even number. As a result, the product $3\,|V_A|$ must also be an even number and, $|V_A|$ must be a multiple of 2 that can be represented as $|V_A| = 2k$. We call k the group class. Substituting the value of $|V_A|$ into (2) the following relations (6) are obtained:

$$|V_A| = 2k, \quad |E_A| = 3k, \tag{6}$$

where k is a natural number. In this manner, different Assur groups are obtained by assigning different values to k. Reference [13] presents a specific algorithm for the synthesis of Assur groups. The simpler Assur group (see Table 2, for $k = 1$) consists of two elements or vertices and three kinematic pairs or edges. The edge connecting vertices 1 and 2 is called *inner*. The edges that connect the group with other Assur groups or the driving-elements group are called *external*.

The class of an Assur group can be defined according to the number k. In this manner, the Assur group with two vertices ($k = 1$) will be called a group of the first class; the group of four vertices ($k = 2$) will be called a group of the second class; the group of six vertices ($k = 3$) will be called a group of the third class and so on. It is also common to define the order r of an Assur group as the number of external edges of its representing graph. It should be noted that two or more Assur groups sharing the same class k may have a different order r as it can be seen in Table 2. The dashed vertices of the Assur groups graph representation do not belong to the Assur groups themselves, but to those groups or driving-elements group they might connect with.

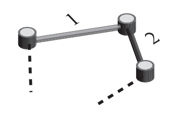
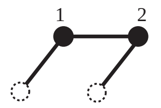
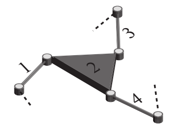
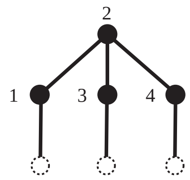
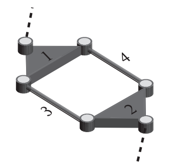
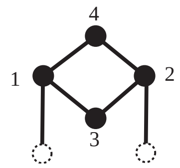
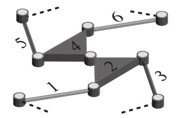
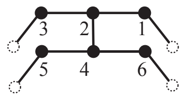
If the class k of an Assur group and its order r are known, then the following information is obtained from (3), (5) and (6):

$|V_A| = 2k$, the number of vertices of the group,
$|E_A| = 3k$, the number of edges of the group,
r, the number of external edges of the group,
$L = k + 1$, the number of independent loops, and
$L_c = k + 1 - r$, the number of closed loops.

3.3 Goal of decomposition

The goal of decomposition of the presented algorithm is formally stated in this section. For this purpose, the required graph operations are first defined and implemented into a mathematical definition of the generation principle of mechanisms.

Table 2 Different types of Assur groups

k	e	v	r	L	L_c	Structural representation	Graph representation
1	3	2	2	2	0		
2	6	4	3	3	0		
2	6	4	2	3	1		
3	9	6	4	4	0		

3.3.1 Graph operations

Merge vertices: a couple of vertices v_i and v_j of a graph G are said to be merged, if the vertices are replaced with a new one, such that every edge incident with v_i or v_j, or with both is incident with the new vertex.

Graph union: let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two sub-graphs of a graph $G = (V, E)$. The union of the sub-graphs G_1 and G_2, $G_1 \cup G_2$, is another sub-graph $G_3 = (V_3, E_3)$ such that $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$. The following properties are applied to the graph union operator $\cup$.

a. Commutative property. This is, $G_1 \cup G_2 = G_2 \cup G_1$.
b. Left-associative property. This is, $G_1 \cup G_2 \cup G_3 = ((G_1 \cup G_2) \cup G_3)$.

Graph subtraction: let $G_A = (V_A, E_A)$ be a sub-graph of a graph $G = (V, E)$. The subtraction of the graphs $G \ominus G_A$ is another sub-graph $G_B = (V_B, E_B)$ such that $V_B = V - V_A$ is the set resulting from deleting all the vertices V_A from V and $E_B = E - E_A$ is the set resulting from deleting all the edges E_A from E. The Left-associative property is applied to the graph subtraction operator $\ominus$. This is, $G_1 \ominus G_2 \ominus G_3 = ((G_1 \ominus G_2) \ominus G_3)$.

3.3.2 Generation principle of mechanisms

Let $G = (V, E)$ be the graph representing a planar mechanism and $G_D = (V_D, E_D)$ be the graph representing the driving-elements group. There exists a sequence of Assur groups $[G_{A1}, G_{A2}, \ldots, G_{An}]$ such that the result of joining the driving-elements group $G_D = (V_D, E_D)$ with the sequence of its Assur groups is G. This sequence represents the kinematic structure of the mechanism and it is the generation principle. It is expressed through the graph union operation as shown in (7):

$$G = G_D \cup \left\{ \bigcup_{i=1}^{n} G_{Ai} \right\}, \tag{7}$$

where n is the number of Assur groups present in the mechanism. Note that the generation principle of a mechanism depends on the selection of its driving-elements group. Furthermore, there are cases in which the sequence is partial since two or more groups share the same hierarchy in the generation principle.

3.4 Algorithm for graph-based generation principle of mechanisms

The result of application of the algorithm 1 to a graph $G = (E, V)$ meeting conditions described in section 3.2.1 is a sequence of Assur groups that satisfies its generation principle (see (7)).

The statement $S = \text{CandidatesForClass}(G, k)$ (algorithm 1, line 11) is a combinatorial function that generates a set S containing all the sub-graphs of G that have a number of vertices is equal to $2k$. The statement $Condition = \text{AssurCondition}(G, G_A)$ (algorithm 1, line 14) is a boolean function that checks whether the graph G_A represents an Assur group or not. Two conditions are verified:

a. If G_A is connected and its number of edges $|E_A|$ equals to $3k$.
b. If $W(G \ominus G_A) = 0$.

If conditions a and b are satisfied, then (4) and (5) are verified and G_A represents a sub-graph (Assur group) of the generation principle.

3.4.1 Running time of the algorithm

The number $|V|$ of vertices of the graph G is a measure of input size. Each of the inner loop statements

Algorithm 1 Decomposition of planar mechanisms into Assur groups

Require: $G = (V, E)$ Graph representing the planar mechanism.
$G_D = (V_D, E_D)$ Sub-graph of G representing the primary-elements groups of G

Ensure: $Assur\_decomposition = [G_{A1}, G_{A2}, \ldots, G_{An}]$ Assur decomposition is a sequence such that the graphs G_{Ai} are sub-graphs of G and $G = G_D \cup \left\{\bigcup_{i=1}^{n} G_{Ai}\right\}$

1: % Merge each of the driving-elements group vertices (V_D) of G and vertex zero
2: $G \leftarrow$ MergeVertices(G, G_D)
3: $Assur\_decomposition \leftarrow [\ ]$
4: % Let Φ be the graph having only vertex zero and no edges
5: **while** $G \neq \Phi$ **do**
6:    % Search starts with Assur class 1
7:    $k \leftarrow 1$
8:    $Assur\_gr\_found \leftarrow$ FALSE
9:    **while** $k \leq \frac{|V|-1}{2}$ **and** Not $Assur\_gr\_found$ **do**
10:       % Generate a set of sub-graphs of G candidates for being Assur groups of the class k
11:       $S \leftarrow$ CandidatesForClass(G, k)
12:       **while** S **and** Not $Assur\_gr\_found$ **do**
13:          $G_A \leftarrow$ First(S)
14:          **if** AssurCondition(G, G_A) **then**
15:             $Assur\_gr\_found \leftarrow$ TRUE
16:             % Update the generation principle
17:             $Assur\_decomposition \leftarrow [Assur\_decomposition, G_A]$
18:             % Update graph G, the symbol $\ominus$ stands for graph subtraction
19:             $G \leftarrow G \ominus G_A$
20:          **else**
21:             % Remove G_A from the set S
22:             $S \leftarrow S - G_A$
23:          **end if**
24:       **end while**
25:       $k \leftarrow k + 1$
26:    **end while**
27: **end while**

12 - 24, including the conditional and looping statements, requires a constant amount of time $O(1)$. By the sum rule, the combined running time of this group of statements is $O(\max(O_{12}, \ldots, O_{24})) = O(1)$.

In the next steps, we assume the worst-case scenario complexities. The number of iterations of the loop of lines 12 - 24 is assumed to be $|S|$. By the product rule, the time spent in this loop is $O(|S| \times 1)$ which is $O(|S|)$. S is the set of sub-graphs of G having a number of vertices equal to $2k$ and its size $|S|$ is given by (8):

$$|S| = \frac{(|V|-1)!}{(2k)!(|V|-2k-1)!}, \tag{8}$$

where the term $(|V|-1)$ corresponds to the number of vertices of G when vertex zero is not included.

Regarding the loop of lines 9 - 26, statement 11 generates the set of sub-graphs of G having a number of vertices equal to $2k$ and therefore, it requires an amount of time $O(|S|)$. Statement 25 requires an amount of time equal to $O(1)$. Finally, statements 12 - 24, corresponding to the inner loop, require an amount of time $O(|S|)$, as stated before. By the sum rule, the combined running time of this group of statements is $O(\max(O_9, \ldots, O_{26})) = O(|S|)$, which is the total amount of time the loop body takes for each iteration. Hence, the number of operations of the loop is given by (9):

$$\begin{aligned} \sum_{k=1}^{\frac{|V|-1}{2}} |S| &= \sum_{k=1}^{\frac{|V|-1}{2}} \frac{(|V|-1)!}{(2k)!(|V|-2k-1)!} \\ &= \sum_{k=1}^{\frac{|V|-1}{2}} \binom{|V|-1}{2k}. \end{aligned} \tag{9}$$

The total running time of the loop can be bounded above by (10):

$$\sum_{k=0}^{|V|} \binom{|V|}{k}. \tag{10}$$

By the implementation of the binomial theorem of Newton, it can be proved that the total amount of time of the loop (lines 9 - 26), takes a time proportional to the power of the number of vertices ($|V|$) of the graph

G (see (11)):

$$\sum_{k=0}^{|V|} \binom{|V|}{k} = 2^{|V|}. \tag{11}$$

Let us consider the outer loop of lines 5 - 27. Each of the statements 7, 8 takes an amount of time $O(1)$, while statements 9 - 26 take an amount of time $O(2^{|V|})$ as stated in (11). By the sum rule, the combined running time of this group of statements is $O(\max(O_5, \ldots, O_{27})) = O(2^{|V|})$.

The outer loop is executed a number of times equal to the number of Assur groups present in the generation principle of G. Since we are looking for the worst-case running time, we assumed a maximal number of Assur groups forming G. This is, that all groups are assumed to have class $k = 1$ and therefore, each having 2 vertices. According to this, for a total number of free vertices equal to $(|V| - 1)$, the total number of Assur groups of the first class is $(|V| - 1)/2$. By the product rule, the time spend in the outermost loop is $O(2^{|V|} \times (|V| - 1)/2)$, so the total operations number of the program is bounded by $O(|V|\, 2^{|V|})$.

3.5 Proof of correctness

For the outermost loop between lines 5 - 27, the invariant Inv that describes the state of the program when iteration j starts is (12):

$$\{Inv : G_j = (V_j, E_j) \ \wedge \ 3(|V_j| - 1) = 2|E_j|\}, \tag{12}$$

equation (12) means that the mechanism at the start of iteration j is described by the graph G_j and it has zero DOF, or $3(|V_j| - 1) = 2|E_j|$. The term $(|V| - 1)$ results from taking vertex zero out of the count of the DOF of G. The algorithm starts with a graph G representing the mechanism in iteration 1 ($G = G_1$). In each iteration, G_j ($j = 1, ...$) becomes smaller because an Assur group graph G_{Aj} is subtracted from it until eventually G_j equals Φ, the empty graph. The algorithm stops in a finite number of iterations. In each iteration, G_j satisfies the invariant Inv in (12). The decomposition $[G_{A1}, G_{A2}, ...]$ of the original G has been calculated.

Let $G_{Aj} = (V_{Aj}, E_{Aj})$ be the Assur group graph identified in the j-th iteration of the inner loop (algorithm 1, line 12). Being G_{Aj} an Assur group, it satisfies that $3|V_{Aj}| = 2|E_{Aj}|$. The algorithm then subtracts G_{Aj} from G_j, resulting in the new graph $G_{j+1} = G_j \ominus G_{Aj}$ (algorithm 1, line 19). To prove that the loop correctly maintains the invariant Inv, we must show that the mechanism remaining after the subtraction, G_{j+1}, still has zero DOF (i.e. it satisfies $3(|V_{j+1}| - 1) = 2(|E_{j+1}|)$. The proof is in the equations (13) to (15):

$$|V_{j+1}| = |V_j| - |V_{A_j}| \ \Rightarrow \ |V_j| = |V_{j+1}| + |V_{A_j}|, \tag{13}$$

$$|E_{j+1}| = |E_j| - |E_{A_j}| \ \Rightarrow \ |E_j| = |E_{j+1}| + |E_{A_j}|, \tag{14}$$

$$\begin{aligned} &3(|V_j| - 1) = 2|E_j| \ \Rightarrow \\ &3|V_{j+1}| + 3|V_{A_j}| - 3 = 2|E_{j+1}| + 2|E_{A_j}| \ \Rightarrow \\ &3|V_{A_j}| = 2|E_{A_j}| \ \Rightarrow 3(|V_{j+1}| - 1) = 2|E_{j+1}|, \end{aligned} \tag{15}$$

therefore, reaching the invariant for iteration $j + 1$. We have proved that each iteration j preserves the invariant Inv of the loop.

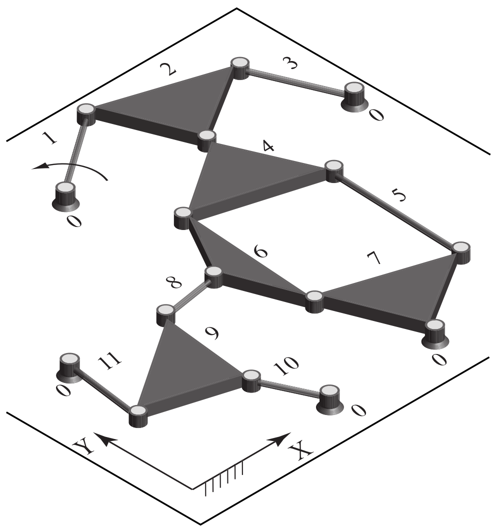
Equations (13) and (14) reflect the fact that the size of the graph G_j decreases in each iteration by exactly the size of the Assur group graph G_{Aj} subtracted from G_j. Equation (15) uses the fact that G_j has zero DOF $(3(|V_j| - 1) = 2|E_j|)$ and the decrement in size of sets V_j and E_j. Equation (15) also uses the Assur character of G_{Aj} $(3|V_{Aj}| = 2|E_{Aj}|)$ to prove that the new graph G_{i+1} has, again, zero DOF. The reader interested in graph subtraction properties may wish to refer to Section 3.3.1.

It is clear that both inner loops are correct. In the loop of statements 5 - 26, the function CandidatesForClass generates a set of Assur candidates with class k. After that, if an Assur group is not found, then the loop increases the value of the class by one. Since k is upper bounded by the number $\left(\frac{|V|-1}{2}\right)$, we guarantee that this loop is executed a finite number of times. As for the loop of statements 12 - 24, the functions First and AssurCondition carry out the extraction and evaluation of each candidate in the set. The loop ends when all candidates have been evaluated or when a candidate graph meets the Assur group condition.

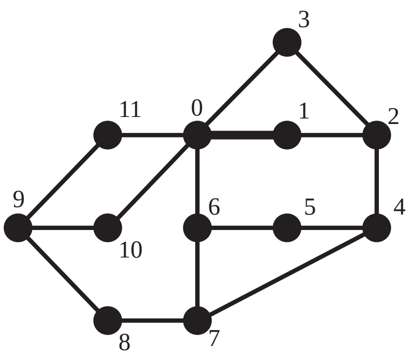
4 Results

In this section, we use algorithm 1 to determine the generation principle of a 12-bar mechanism with one DOF. The mechanism is also used in [1] as a test example of the structural analysis of mechanisms. Figure 4 shows the structural representation of the aforementioned mechanism, together with its representing graph $G = (V, E)$.

The notation S_i represents the set S in the i-th iteration of the loop in lines 9 - 26. The first step in algorithm 1 merges the driving-element group with vertex zero. Figure 5 shows the result of this operation. Now the algorithm starts searching for Assur groups with class k equals to 1. At this point, the function CandidatesForClass generates a set S_1 of candidates G_A for being Assur groups, each having a number of

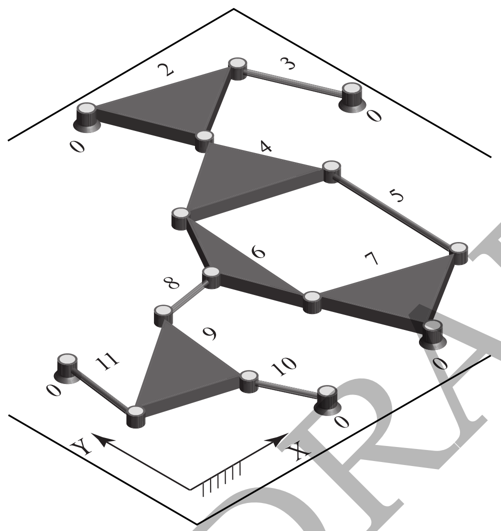


(a) Structural representation

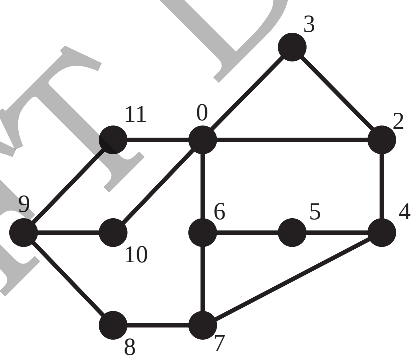


(b) Graph representation of Fig. 4(a)

Fig. 4 12-bar planar mechanism presented in [1]



(a) Structural representation



(b) Graph representation of Fig. 5(a)

Fig. 5 12-bar planar mechanism: Merge of the driving-elements group with vertex zero

vertices $|V_A|$ equal to 2. Table 3 summarizes the set S_1.

The number of candidates of S_1 is given by $\binom{|V|-1}{2k}$ and is equal to 45. The last column of Table 3 corresponds to the output of the function AssurCondition after each candidate G_A is evaluated.

Table 3 Generation-principle analysis of a 12-bar mechanism: Set S_1 of Assur candidates with $k = 1$

G_A	11	10	9	8	7	6	5	4	3	2	$\|V_A\|$	$\|E_A\|$	AC$^{(a)}$
1	0	0	0	0	0	0	0	0	1	1	2	4	F
2	0	0	0	0	0	0	0	1	0	1	2	5	F
3	0	0	0	0	0	0	0	1	1	0	2	5	F
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
45	1	1	0	0	0	0	0	0	0	0	2	4	F

(a) AssurCondition (AC)

The algorithm increases the value of the Assur class by 1 and starts searching for Assur groups with class $k = 2$, since none of the candidates in S_1 meets the AssurCondition. The new set S_2 of Assur group candidates is generated by the function CandidatesForClass (see Table 4). Each of the candidates in S_2 has a number of vertices equal to four. The last candidate of the set S_2 (Table 4, candidate number 210), consisting of vertices 8, 9, 10 and 11 meets the AssurCondition, and therefore it is an Assur group.

Table 4 Generation-principle analysis of a 12-bar mechanism: Set S_2 of Assur candidates with $k = 2$

G_A	11	10	9	8	7	6	5	4	3	2	$\|V_A\|$	$\|E_A\|$	AC
1	0	0	0	0	0	0	1	1	1	1	4	7	F
2	0	0	0	0	0	1	0	1	1	1	4	9	F
3	0	0	0	0	0	1	1	0	1	1	4	8	F
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	0	0	0	0	1	1	1	1	0	0	4	7	F
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
210	1	1	1	1	0	0	0	0	0	0	4	6	T

The Assur group found in line 15 is subtracted from the graph G and the algorithm starts again searching for Assur groups with class $k = 1$. The function CandidatesForClass generates a new set S_3 of Assur group candidates. However, none of the candidates of S_3 meets the AssurCondition and the Assur class value increases by 1. Therefore, the algorithm searches Assur groups with class $k = 2$.

A set S_4 of Assur group candidates with class two is generated by the function CandidatesForClass. In this case, candidate number 15 presented in Table 5, consisting of vertices 4, 5, 6 and 7, meets the AssurCondition and it is subtracted from the graph G.

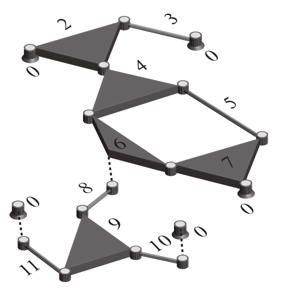
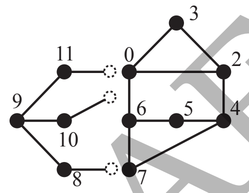
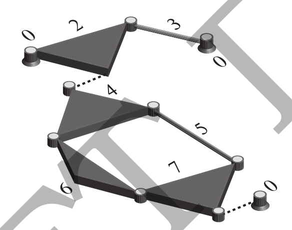
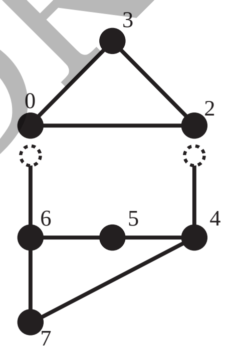
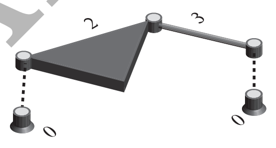
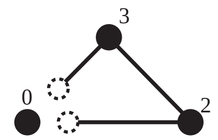
The procedure is iterated until all the Assur groups are subtracted from G, which eventually becomes the trivial graph Φ. Table 6 illustrates the progress of the algorithm towards termination.

Table 5 Generation-principle analysis of a 12-bar mechanism: Set S_4 of Assur candidates with $k = 2$

G_A	7	6	5	4	3	2	$\|V_A\|$	$\|E_A\|$	AC
1	0	0	1	1	1	1	4	7	FALSE
2	0	1	0	1	1	1	4	9	FALSE
3	0	1	1	0	1	1	4	8	FALSE
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	1	1	1	1	0	0	4	6	TRUE

The output of the algorithm is a sequence consisting of three Assur groups. The first two elements have class $k = 2$ meanwhile the last one has class $k = 1$. The generation principle of this mechanism is obtained through the graph union of such groups starting from the last founded group and ending with the first one.

Table 6 Generation principle of a 12-bar mechanism: Summary of the analysis by means of algorithm 1

i	Structural representation	Graph representation
3		
2		
1		

5 Conclusions and Future Work

A new graph-based algorithm for the decomposition of planar mechanisms into Assur groups has been implemented. Unlike other procedures, neither previous data manipulation nor visual inspection is required. In this manner, the algorithm can be systematically applied to complex mechanisms. The number of operations of the algorithm execution is bounded by $O(|V|\,2^{|V|})$, where $|V|$ is the number of links. Since in actual mechanisms the number of links is rarely numerous, then the execution stands for a realistic solution.

Applying the algorithm to a graph not only provides information about the Assur groups within its representing mechanism, but also about the sequence in which those groups are connected. This is useful when attempting to perform a modular approach for the kinematic and force analysis of the mechanism.

Future work will address the application of the proposed algorithm to planar mechanisms with floating input links.

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