Using Gröbner Bases in Kinematic Analysis of Mechanisms

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Introduction

The Geometric Constraint Satisfaction or Scene Feasibility (GCS / SF) problem consists of a basic scenario containing geometric entities, whose context is used to propose constraining relations among still undefined entities. If the constraint specification is consistent, the answer to the problem is one of finitely or infinitely many solution scenarios satisfying the prescribed constraints. Otherwise, a diagnostic of inconsistency is expected. The mathematical approach, previously presented in other publications, describes the problem using a set of polynomial equations, with the common roots to this set of polynomials characterizing the solution space for such a problem. That work presents the use of Gröbner basis techniques for assessing consistency and redundancy of the constraints. It also integrates subgroups of the Special Euclidean Group of Displacements $SE(3)$ in the problem formulation to exploit the structure implied by geometric relations. In this article, the application of the discussed techniques to kinematic analysis of mechanisms is illustrated by an example. It is implemented using MAPLE's routines to manipulate polynomial ideals, and calculate their Gröbner Bases.

GCS / SF underlies a number of problems in CAM / CAM / CAPP areas, for example, *fixturing, assembly planning, constraint-based design, tolerancing and dimensioning, kinematic analysis*, etc. Therefore, it is evident that a strong theoretical and practical background satisfying geometric constraints is crucial in CAD / CAM / CAPP.

Topology and *Geometry* are two interdependent aspects of GCS / SF although they have often been treated independently. Topology deals exclusively with the connectivity and nature of the spatial relations between entities. Geometry refers to the distances and directions that parameterize these relationships. Topologically, this work will address constraints which can be expressed as algebraic equalities [10]. Geometrically, it is restricted to zero curvature (points, straight lines and planes) proper subsets of E^3.

Literature Survey

Solving GCS / SF implies the ability to: (i) instance entities (or produce configurations) which satisfy the given constraints, (ii) identify a redundant constraint, (iii) identify an inconsistent set of constraints, and (iv) determine the degrees of freedom between several entities. These requirements immediately preclude the application of purely numerical techniques. In current literature, GCS / SF has been approached from the areas of group theory (Herve [5]) and kinematics and mechanisms (Angeles [2]). Therefore, the terms *(trivial) constraint, joint* and *group* are used interchangeably in the discussion.

Popplestone *et al* [1, 9] formalized GCS / SF in the form of equations of unknown positioning matrices. They also explored the application of finite groups in situations involving symmetries such as arrays, hexagonal pieces, mirror arrangements, etc.

Based on Herve's formalization, Thomas, Torras and Celaya in [12] attempted the topological reduction of constraint networks. Limitations of this work are the topology-only treatment, and the type of constraints (trivial) that it considers. Its contribution is the methodology proposed to state the (geometry and topology of) GCS / SF in terms of the $SE(3)$ group.

In [10], Ruiz & Ferreira formulated the GCS / SF problem as one of determining the solution space of a set of polynomials, using Gröbner Bases to characterize the solution space using [7]. The method allowed for the integration of geometric and topological reasoning, and the efficient group theoretic formulation of Herve.

This article illustrates how Algebraic Geometry ([3, 6, 7]) and Group Theory ([5, 12, 2]) can be applied to the specific area of Kinematic Analysis. For that purpose, section 2 presents a summary of the necessary background. Section 3 directly relates GCS / SF to Kinematic Analysis. Section 4 discusses the specific example of the Oldham mechanism. Section 5 briefly presents relevant conclusions, while Appendix A presents the MAPLE script used to analyze the Oldham coupling.

Background

Algebraic Geometry and the GCS / SF Problem

The GCS / SF problem takes place in a world W, with a set of relations R. If a set of entities $S = \{e_1, .., e_n\}$ satisfies the constraints, it is said that *S is feasible for W and R*, and this fact is written as $S = feasible(W, R)$. If the polynomial form of the problem is $F = \{f_1, f_2, ..., f_n\}$ with f_i polynomials in variables $x_1, x_2, ..., x_n$, it is said that $F = poly\_form\ (W, R)$. Since S is a solution for F, it is denoted as $S = solution(F)$.

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For the purposes of this paper, the calculation of the Gröbner Basis of a set of polynomials F can be regarded as a black box procedure whose result, $GB(F)$, an alternative set of polynomials, has several important properties. The properties allow us to draw the following propositions ([6, 7]):

1. $S = solution(F)$ iff $S = solution(GB(F))$.
   In the context of GCS / SF, this implies that $GB(F)$ and F describe the same scene, although $GB(F)$ presents properties useful in the solution process.

2. $1 \in GB(F) \Rightarrow S = solution(F) = \Phi$
   This property implies that finding "1" or a constant polynomial in $GB(F)$ implies the equation "0=1" leading to the fact that F has no solution.

3. F is *Zero-dimensional* iff F (and $GB(F)$) has a finite number of solutions. The zero-dimensionality of I can be assessed: A variable x is non-instanced if it does not appear as $head(p)$ for any polynomial $p \in GB(F)$ ($p = x^d + tail(p),\ d \in N$).

4. Let a new constraint be represented by polynomial f. f is redundant to $F \Leftrightarrow (1 \in GB(F \cup \{y.f - 1\}))$ for a new variable y. It establishes that the satisfaction of the new constraint f is unavoidable when the initial set of constraints is satisfied.

5. $GB(F)$ (based on a lexicographic order $x_1 \succ x_2 \succ x_3 ... \succ x_n$) is a triangular set in the sense that $GB(F)$ contains polynomials only in x_1, some others only in x_1, x_2, and so on, making the numerical solution a process similar to triangular elimination.

The theoretical framework for the solution of GCS / SF can be summarized in the following procedure [10]: In the event of the addition of new constraints to the scene, they are converted into polynomial(s), tested for redundancy (Proposition 4), consistency (Proposition 2) and finiteness of number of solutions (Proposition 3). If the new constraint is redundant, it is ignored (Proposition 4). If the ideal has become zero-dimensional a triangular Gröbner Basis under a stated lexicographic order is extracted and solved (Proposition 5). Proposition 1 is the underlying basis of the procedure, since it establishes that the $GB(F)$ faithfully represents F.

The use of a group-theoretical approach to express GCS / SF is explored next. It will provide (i) a direct relation between degrees of freedom and variables, and (ii) a smaller formulation of the problem.

Group-Theoretic Formulation for the GCS / SF Problem

This section examines the modeling of GCS / SF by using the canonical form of conjugation classes developed by Herve [5] and the application of his work by several authors (refs [2, 10, 12]). The set of Euclidean displacements in 3D, $SE(3)$, is a (non commutative) group [8] with the composition operation ($\circ$). $SE(3)$ presents subsets which are groups themselves, and which express certain common classes of displacements. They are called *subgroups*. For example, the subgroup of the rotations about a *given* axis u in the space, R_u, is a subset of $SE(3)$, and a group itself. A list of the subgroups of $SE(3)$ and their canonical representation [5], as well as their degrees of freedom is shown in Table 1 [1]. For example, "rotations" are all transformations of the form

$$Ru(\theta) = B.\hat{R}_u.B^{-1} = B.twix(\theta).B^{-1}.$$

The displacement $B \in SE(3)$ represents the *geometric* part of a particular constraint, while the canonical part ($\hat{R}_u$) contains the *topological* information.

Using this methodology, the contact constraints appear in Table 2. For example, a $P - ON - PLN$ relation confines a point to be on a plane, therefore configuring a 5-dof constraint [2]. These (matrix) equations produce the polynomial form of GCS / SF.

GCS/SF is stated as a series of constraints R_i relating F_{i1} with F_{i2} as shown in Figure 1 (corresponding to a two body system), where F_{ij} is the i^{th} feature of body B_j. The $R_i()$ constraints are as dictated by Tables 1 and 2. Body B_1 (in this case) contains two features, F_{11} and F_{21}. B_2 contains F_{12} and F_{22}. The goal is to find a final position of B_1 (assuming B_2 stationary), such that F_{11} relates to F_{12} and F_{21} relates to F_{22} satisfying the invariance dictated by $R_1()$ and $R_2()$ respectively.

The equations expressing the facts above are:

$$B_1.F_{11}.R_1() = B_2.F_{12} \ : \ B_1.F_{21}.R_2() = B_2.F_{22} \qquad (1)$$

The above procedure can be generalized to larger systems. The solution and interpretation of Equation 1 follow the constraint management procedure discussed in the last section [10, 5, 10].

The GCS / SF Problem in Design and Analysis of Mechanisms

GCS / SF and Kinematic Analysis of Mechanisms are related since kinematic joints are constraints on the relative position of entities. A solution for GCS / SF is a physically realizable configuration of kinematic links. Continuous regions of the solution space for GCS / SF directly map to the possible motion (degrees of freedom) of the mechanism.

[1] In this table, $twix(\theta)$ means a rotation about the X axis by θ. $XTOY$ means a rotation by 90^0 about the Z axis. $trans(x, y, z)$ indicates a general spatial translation.

[2] Points are in the origin of their attached frame. Lines coincide with the X axis of their frame. Planes coincide with the Y-Z plane of their attached frame.

Table 1: Conjugation Classes and their Canonical Forms

Dof	*Symbol*	*Conjugation Class*	*Canonical Subgroup*
1	R_u	Rotations about axis u	$\{twix(\theta)\}$
1	T_u	Translations along axis u	$\{trans(x, 0, 0)\}$
1	$H_{u,p}$	Screw movement along axis u, with pitch p	$\{trans(x, 0, 0).twix(px)\}$
2	C_u	Cylindrical movement along axis u	$\{trans(x, 0, 0).twix(\theta)\}$
2	T_p	Planar translation parallel to plane P	$\{trans(0, y, z)\}$
3	G_p	planar sliding along plane P	$\{trans(0, y, z).twix(\theta)\}$
3	S_o	Spherical rotation about center "o"	$\{twix(\psi).XTOY.twix(\phi).XTOY.twix(\theta)\}$
3	T	3D translation	$\{trans(x, y, z)\}$
3	Y_v,p	Translating Screw axis v, pitch p	$\{trans(x, y, z).twix(px)\}$
4	X_v	3D translation followed by rotation about v	$\{trans(x, y, z).twix(\theta)\}$

Table 2: Entity Relations in the Form of Kinematic Joints

macro	*joint chain*	*kinematic joints in chain*	*dof*
P-ON-P	S	spherical	3
P-ON-LN	$T_v \circ S_o$	linear translation, spherical	4
P-ON-PLN	$T_P \circ S_o$	planar translation, spherical	5
LN-ON-LN	C	cylindrical	2
LN-ON-PLN	$T_P \circ R_v \circ R_w$	planar translation, revolute	4
PLN-ON-PLN	$T_P \circ R_v$	planar translation, revolute	3

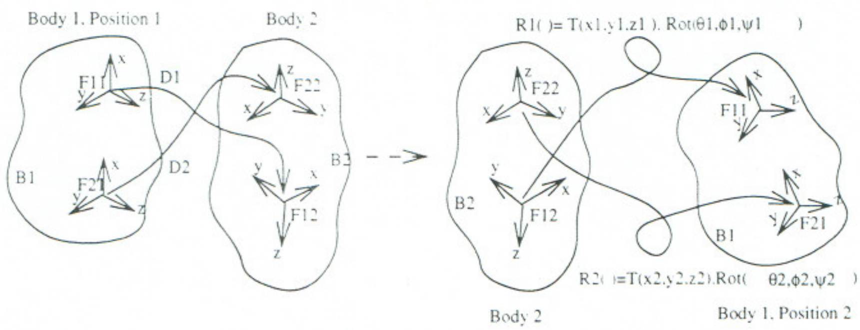

Figure 1: Two Body Example of Canonical Variable Modeling of the GCS/SF Problem

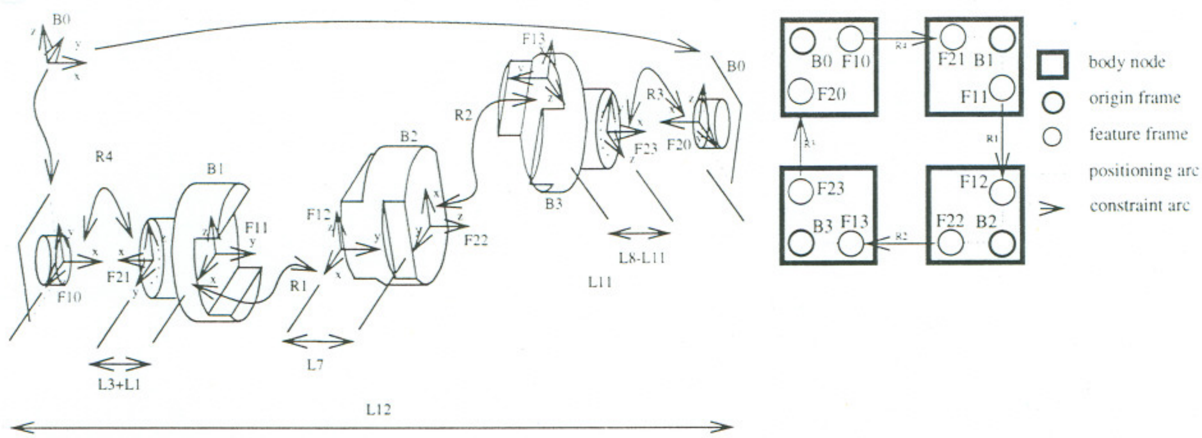

Figure 2: Piece Disassembly and Spatial Constraint Graph of Oldham Mechanism

The Spatial Constraint (SC) graph (see Figure 2), conveys the topological and geometrical information of GCS / SF. It is suitable for the computer generation of the equations governing the scene and allows the identification of subproblems which help in GCS / SF by allowing the application of preprocessing techniques [11].

Conventions: Since entities are represented by frames, the terms *entity* and *frame* are equivalent. In the SC graph the nodes are *entity* frames (B_j and F_{ij}). The arc between two nodes represents the displacement that relate the corresponding entity frames. There are three types of nodes: nodes B_j which represent the origin frame of a body in the World Coordinate System, feature nodes F_{ij} which represent the feature i in body B_j and *body* nodes include the origin frame of the body and its features. Conceptually, there are two types of arcs: *positioning* and *constraint* arcs. Positioning arcs represent *known* relative positions of features within bodies. They always join an entity B_i and one of its features F_{ji}. Constraint arcs always connects two feature nodes, which may be joined by more than one arc to admit more than one constraint between them. The constraint arcs are represented by $C_i(x_j, \theta_m, ..)$, with the degrees of freedom $x_j, \theta_m ...$ sometimes being omitted. To simplify the notation, positioning arcs are named F_{ji}, as the features themselves, and the body nodes are named as their origin frame, B_j.

The Oldham Coupling

The Oldham coupling is shown in Figure 2. This mechanism is designed to connect two parallel, non-collinear axes, allowing the transmission of rotational movement [4]. Using canonical variables, the types of joints present in this mechanism are modeled in Table 3. The ground frame is called B_0, and supports the Oldham coupling through the features F_{20} and F_{10} which are the parallel, non-collinear axes. The two central joints R_1 and R_2 are prismatic, with their grooves F_{12} and F_{22} being non-parallel. R_3 and R_4 represent the rotatory movements (input / output) that are to be transmitted by the coupling.

Table 3: Joint List of the Oldham Coupling

Joint	*Joint Type*	*Canonical Representation*
R_1	Tu	$trans(x_1)$
R_2	Tu	$trans(x_2)$
R_3	Ru	$twix(\theta_3)$
R_4	Ru	$twix(\theta_4)$

Modeling of Oldham Coupling with Canonical Variables

Using the methodology developed in [10] and the SC graph (Figure 2), the kinematic relations may be expressed by the following matrix equations:

$$\begin{aligned} &B_1.F_{11}.R_1.F_{12}^{-1} = B_2 \quad B_2.F_{22}.R_2.F_{13}^{-1} = B_3 \\ &B_3.F_{23}.R_3.F_{20}^{-1} = B_0 \quad B_0.F_{10}.R_4.F_{21}^{-1} = B_1 \end{aligned} \tag{2}$$

In this case the graph of constraints contains only one loop and it can be expressed by the equation:

$$\begin{aligned} &F_{10}.R_4.F_{21}^{-1}.F_{11}.R_1.F_{12}^{-1} \\ &= F_{20}.R_3.F_{23}^{-1}.F_{13}.R_2.F_{22}^{-1} \end{aligned} \tag{3}$$

where I_4 is the 4×4 identity matrix. This equation conveys the topological configuration of the coupling.

In the Oldham coupling, the non-collinearity between axes F_{20} and F_{10} is expressed by L_3. The total length of the mechanism is determined by parameter L_{12}. According to Equation 3 and Table 3, the polynomials which express the kinematics of the mechanism are:

$$\begin{gathered} -L_{12} + L_8 - L_{11} + L_7 + L_1 + L_3 = 0 \\ s\theta_4\, s\theta_3 - c\theta_4\, c\theta_3 - 1 = 0 \\ -s\theta_4\, c\theta_3 - c\theta_4\, s\theta_3 = 0 \\ L_3\, s\theta_4\, c\theta_3 + L_3\, c\theta_4\, s\theta_3 - s\theta_4\, x_2 + c\theta_4\, x_1 = 0 \\ c\theta_4\, s\theta_3 + s\theta_4\, c\theta_3 = 0 \\ s\theta_4\, s\theta_3 - c\theta_4\, c\theta_3 - 1 = 0 \\ -L_3\, s\theta_4\, s\theta_3 + L_3\, c\theta_4\, c\theta_3 - c\theta_4\, x_2 - s\theta_4\, x_1 = 0 \\ s\theta_3^{\,2} + c\theta_3^{\,2} - 1 = 0 \\ s\theta_4^{\,2} + c\theta_4^{\,2} - 1 = 0 \end{gathered} \tag{4}$$

Under the specification presented, the analysis of the Oldham coupling should predict the transmission of non-collinear, rotatory movement. In a first approximation, however, the Gröbner Basis of this set of polynomials happens to be $GB = \{1\}$. This implies a topological or geometrical inconsistency. A careful examination of Equations 4 reveals that in the first equation, L_{12} is not an *independent parameter* (see Figure 2). This equation indicates that

$$L_{12} = L_8 - L_{11} + L_7 + L_1 + L_3 \tag{5}$$

is a **necessary** condition for the mechanism to be realizable. This detection of geometrical inconsistencies in relation to the prescribed topology is an unexpected bonus of using algebraic geometry techniques for GCS / SF. Under the condition imposed by Equation 5, the first equation in (4) would become $0 = 0$. The lexicographic Gröbner Basis, calculated under the order $x_1 \succ x_2 \succ s\theta_3 \succ c\theta_3 \succ s\theta_1 \succ c\theta_4$, is:

$$\begin{gathered} \underline{x_1} + L_3\, s\theta_4 = 0 \qquad \underline{x_2} + L_3\, c\theta_4 = 0 \qquad \underline{s\theta_3} - s\theta_4 = 0 \\ \underline{c\theta_3} + c\theta_4 = 0 \quad \underline{s\theta_4}^2 + c\theta_4^{\,2} - 1 = 0. \end{gathered} \tag{6}$$

This triangularized Gröbner Basis presents a free variable, $c\theta_4$, responsible for the one-dimensionality of the polynomial ideal [3, 7]. As expected, the prismatic joints $R1$ and $R2$ (variables x_1 and x_2) are controlled by the separation between the two axes, L_{12}.

Variation 1. Cylindrical Joints in Central Connector

A question arising from the previous section is whether other configurations different from the Oldham coupling would also transfer rotational movement between non-collinear axes (see Figure 3). One of such variations is achieved by replacing joints R_1 and R_2, which originally are prismatic (Tu), by cylindrical ones ($Cu = trans(x,0,0).twix(\theta)$). The translational dof x is essential to the functioning of the coupling. It would be expected that the rotational degree of freedom θ be instanced.

Using the cycle formulation from Equation 3, and the joint specification of Table 4, the kinematic equations can be formulated. Under the ordering $x_1 \succ s\theta_1 \succ c\theta_1 \succ x_2 \succ s\theta_2 \succ c\theta_2 \succ s\theta_3 \succ c\theta_3 \succ s\theta_4 \succ c\theta_4$ this equation set has the following lexicographic Gröbner Basis:

$$\begin{gathered} \underline{x_1} + L_3\, s\theta_4 = 0 \qquad \underline{s\theta_1} = 0 \qquad \underline{c\theta_1} - 1 = 0 \\ \underline{x_2} + L_3\, c\theta_4 = 0 \qquad \underline{s\theta_2} + 1 = 0 \qquad \underline{c\theta_2} = 0 \\ \underline{s\theta_3} - s\theta_4 = 0 \quad \underline{c\theta_3} + c\theta_4 = 0 \quad \underline{s\theta_4}^2 + c\theta_4^{\,2} - 1 = 0 \end{gathered} \tag{7}$$

The variable $c\theta_4$, the angular input/output of the mechanism is effectively *the* degree of freedom left. All other variables appear as $head()$ of some polynomial. As expected, the angular freedoms given to the central joints R_1 and R_2 do not affect the degrees of freedom of the whole coupling, since they are immediately instanced. Therefore, the joints act as prismatic rather than cylindrical ones.

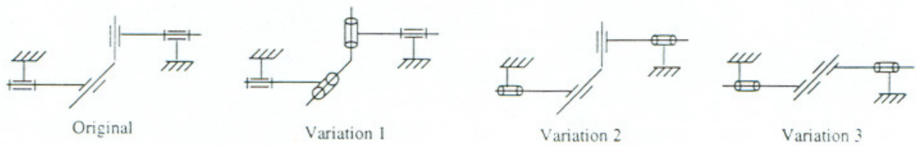

Figure 3: Variations of the Oldham Mechanism

Table 4: Joint List of the Oldham Coupling. Variation 1

Joint	*Joint Type*	*Canonical Representation*
R_1	Cu	$trans(x_1).twix(\theta_1\ )$
R_2	Cu	$trans(x_2).twix(\theta_1\ )$
R_3	Ru	$twix(\theta_3)$
R_4	Ru	$twix(\theta_4)$

Variation 2. Cylindrical Joints in Input / Output Links

If R_3 and R_4 are strictly rotational joints Ru, L_{12} was found dependent on other dimensions. Therefore, by allowing axial movement in joints R_3 and R_4 (see Figure 3), the expected result should be that the L_{12} can be considered an independent parameter. The coupling should act as a transmitter of cylindrical movement along two non-collinear parallel axes.

The constraint polynomials are built under the joint configuration of Table 5 and the matrix Equation 3. Their Gröbner Basis under the ordering $x_1 \succ x_2 \succ x_3 \succ s\theta_3 \succ c\theta_3 \succ x_4 \succ s\theta_4 \succ c\theta_4$ is

$$\begin{aligned}
\underline{x_1} + L_3\, s\theta_4 &= 0\\
\underline{x_2} + L_3\, c\theta_4 &= 0\\
\underline{x_3} + x_4 + L_{12} - L_8 + L_{11} - L_7 - L_1 - L_3 &= 0\\
\underline{s\theta_3} - s\theta_4 &= 0\\
\underline{c\theta_3} + c\theta_4 &= 0\\
\underline{s\theta_4}^2 + c\theta_4{}^2 - 1 &= 0
\end{aligned} \tag{8}$$

This Gröbner Basis represents a two-dimensional ideal with two free variables; x_4 and c_4. c_4 is the rotational movement transmitted. x_4 represents the translational degree of freedom. Variables x_4 and x_3 act as *slack* variables; they allow $L_{12} \neq L_8 - L_{11} + L_7 + L_1 + L_3$, in contrast with the original Oldham coupling in which such condition would render the mechanism unrealizable.

Variation 3. Parallel Prismatic Joints in Central Link

In this variation the *topology* of the Variant 2 (expressed in Table 5) is maintained while the *geometry* is modified in such a way that the solution space changes radically. The features F_{12} and F_{22}, the grooves of the prismatic joints R_1 and R_2, are made parallel. This modification (see Figure 3) would preclude the whole joint for transmitting rotational movement through the non-aligned axis F_{10} and F_{20}. Under the order $x_1 \succ x_2 \succ s\theta_3 \succ c\theta_3 \succ s\theta_4 \succ c\theta_4$ the lexicographic Gröbner Basis of this arrangement would be

$$\begin{gathered}
\underline{x_1} + x_2 + L_3\, s\theta_4 = 0 \quad \underline{s\theta_3} = 0 \quad \underline{c\theta_3} + s\theta_4 = 0\\
\underline{s\theta_4}^2 - 1 = 0 \quad \underline{c\theta_4} = 0
\end{gathered} \tag{9}$$

This triangular Gröbner Basis indicates that all the *angular* variables are locked (instanced). The mechanism cannot transmit rotatory movement, and x_2, the translational variable, is the only degree of freedom left.

Conclusions

It has been shown that the kinematics of mechanical link arrangements can be expressed in terms of GCS / SF. Therefore, the methods of solution for such a problem can be applied in order to analyze the characteristics of kinematic chains.

By using the set of canonical variables, a direct map between the kinematic characteristics of the mechanism and the algebraic expression of the corresponding GCS / SF problem can be established. This direct map can be taken advantage of in inferring possible variations of the mechanism by elaborating about variations on the basis of the polynomial ideal.

Gröbner Bases for Oldham Coupling Example

This section shows the basic calculation for the original Oldham Coupling. The three variations are obvious modifications.

```
# File:          oldham_basic.mapl
# Bodies:0. Ground
```

Table 5: Joint List of the Oldham Coupling. Variation 2

Joint	*Joint Type*	*Canonical Representation*
R_1	Tu	$trans(x_1)$
R_2	Tu	$trans(x_2)$
R_3	Cu	$trans(x_3).twix(\theta_3)$
R_4	Cu	$trans(x_4).twix(\theta_4)$

```
#       1. Rotatory Groove Left
#       2. Central Piece
#       3. Rotatory Groove Right
#
# Constraints as per Tables 1 and 2:
# (R1:Tu F11-F14)   (R2:Tu F21-F14)
# (R3:Ru F12-F15)   (R4:Ru F22-F15)
#==================================
```

First define `twix()`, `trans()`, `xTOy`, `rotX_90`, and `I_4` in order to express Herve's constraints.

```
> twix := proc(s,c)
> RETURN( array([[1,0,0,0],[0,c,-s,0],
>                [0,s,c,0],[0,0,0,1]]))
> end:
> trans := proc(x,y,z)
> RETURN(array([[1,0,0,x],[0,1,0,y],
>                [0,0,1,z],[0,0,0,1]]))
> end:
> xTOy:=array([[0,-1,0,0],[1,0,0,0],
>               [0,0,1,0],[0,0,0,1]]);

> I_4:=array([[1,0,0,0],[0,1,0,0],
>              [0,0,1,0],[0,0,0,1]]);

> rotX_90 := twix(-1,0);

> MirrorX:=array([[-1,0,0,0],[0,1,0,0],
>               [0,0,-1,0],[0,0,0,1]]);
```

Next, simulate the Oldham mechanism.

```
> oldham := proc()
> with( grobner );
> with( linalg );
```

Here, the GEOMETRY is defined.

```
> # Dimensional Parameters:
> L12:= L8 - L11 + L7 + L1 + L3;
> F11:=array([[0,1,0,L1],[-1,0,0,L4],
>              [0,0,1,L2],[0,0,0,1]]);
> F21:=array([[-1,0,0,-L3],[0,-1,0,L4],
>              [0,0,1,L2],[0,0,0,1]]);
> F12:=array([[0,1,0,-L7],[-1,0,0,L5],
>              [0,0,1,L6],[0,0,0,1]]);
> F22:=array([[0,0,1,0],[0,-1,0,L5],
>              [1,0,0,L6],[0,0,0,1]]);
> F13:=array([[0,-1,0,L11],[0,0,-1,L9],
>              [1,0,0,L10],[0,0,0,1]]);
> F23:=array([[1,0,0,L8],[0,0,-1,L9],
>              [0,1,0,L10],[0,0,0,1]]);
> F20:=array([[-1,0,0,L12],[0,0,1,0],
>              [0,1,0,L3],[0,0,0,1]]);
> F10:=array([[1,0,0,0],[0,1,0,0],
>              [0,0,1,0],[0,0,0,1]]);
```

```
> #----------------------------------
> # B0, B1, B2, B3 are immaterial
> # C1:Tu=F11-F12     C2:Tu=F22-F13
> C1 := evalm( trans( x_1 ,0 ,0 ));
> C2 := evalm( trans( x_2 ,0 ,0 ));
> # C3:Ru=F23-F20
> # C4:Ru=F10-F21
> C3 := evalm( twix( s3, c3 ) );
> C4 := evalm( twix( s4, c4 ) );
```

Next, the cycle equation. `MirrorX` and `rotX_90` are finite symmetries. Their handling is out of our scope.

```
> cycle:=evalm(
>   F10&*MirrorX&*C4&*inverse(F21)&*F11&*
>   C1&*inverse(F12)&*F22&*rotX_90&*C2&*
>   inverse(F13) &*
>   F23&*MirrorX&*C3&*inverse(F20)-I_4
>          );
```

Now, separate the equations.

```
> eqs :=  [
>     cycle[1,1],cycle[1,2],cycle[1,3],
>     cycle[1,4],cycle[2,1],cycle[2,2],
>     cycle[2,3],cycle[2,4],cycle[3,1],
>     cycle[3,2],cycle[3,3],cycle[3,4],
>     s3^2+c3^2-1,s4^2+c4^2-1
>         ];
> N := nops( eqs );
> vars := [x_1, x_2, s3, c3, s4, c4 ];
```

Do the calculations and measure performance:

```
> t1 := time();
> GB := gbasis( eqs, vars , plex );
> t2 := time();
> time := t2-t1;
```

Display the results:

```
> #---------------------------------
> print(`Equations:________________`);
> for eq in eqs do print( eq ); od;
> print(`__________________________`);
> print(`Oldham example: `);
> print(`num vars  :  ` , nops( vars ) );
> print(`num eqs   :  ` , nops( eqs ) );
> print(`num eqs GB :  ` , nops( GB ) );
> print(`time GB:  ` ,time );
> print(`__________________________`);
> print(`GBasis:`);
> for g_pol in GB do print(g_pol); od;
> end:
```

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